Symmetric Instability

Brian Fiedler

March 25, 1998

1 Linear Theory

Here we consider the stability of a steady zonal current in thermal wind balance; the basic state represents an idealized "jet stream" in the atmosphere. We consider one special mode of instability, one which looks like "rolls" aligned with the zonal current. For somewhat obscure reasons, this mode is called symmetric instability. The basic state consists of :

$$\vec{U} = u_o(z)\hat{i}. \tag{1}$$

We assume that $\partial/\partial x$ of any quantity is zero (hence "symmetric"?). We choose to analyze this instability with the Boussinesq approximation. The linearized model equations are:

$$\frac{\partial b'}{\partial t} = -N^2 w' + f\Lambda v', \tag{2}$$

$$\frac{\partial u'}{\partial t} = -\Lambda w' + f v', \tag{3}$$

$$\frac{\partial v'}{\partial t} = -f u' - \frac{\partial \varphi'}{\partial y}, \tag{4}$$

$$\frac{\partial w'}{\partial t} = b' - \frac{\partial \varphi'}{\partial z}, \tag{5}$$

and

$$\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{6}$$

where we have written

$$\Lambda \equiv \frac{du_o}{dz}. \tag{7}$$

For the purpose of making mathematical analysis possible with constant coefficient equations, we will assume that f, Λ and N^2 are constants. Unlike the shear flow analyzed in *An Unstable Parallel Flow*, it turns out that we do not risk losing an important aspect of the instability by working with constant Λ. We propose a solution of form

$$(u', v', w', \varphi', b') = \left(\hat{u}, \hat{v}, \hat{w}, \hat{\varphi}, \hat{b}\right) e^{i(ly - \omega t)} \tag{8}$$

where the $\widehat{\ }$ quantities depend on z. Equations (2)-(6) become:

$$-i\omega\hat{b} = -N^2\hat{w} + f\Lambda\hat{v} \tag{9}$$

$$-i\omega\hat{u} = -\Lambda\hat{w} + f\hat{v} \tag{10}$$

$$-i\omega\hat{v} = -f\hat{u} - il\hat{\varphi} \tag{11}$$

$$-i\omega\hat{w} = \hat{b} - \frac{d\hat{\varphi}}{dz} \tag{12}$$

$$il\hat{v} + \frac{d\hat{w}}{dz} = 0. \tag{13}$$

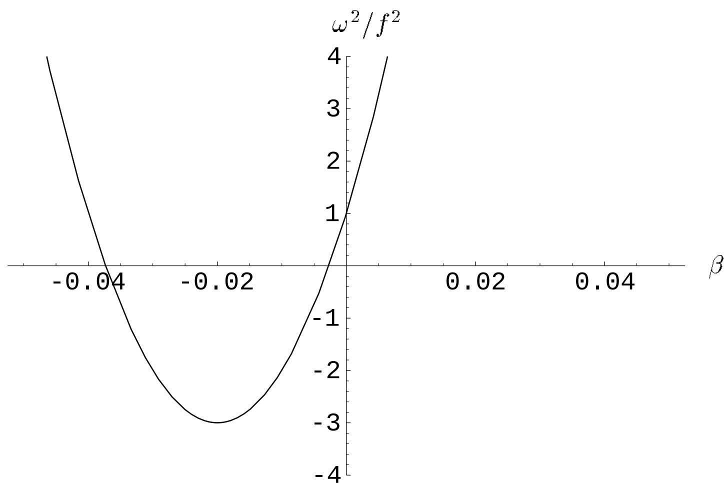

Figure 1: ω^2 from (18) with $N = 100f$ and $\Lambda = 2N$.

Equations (9) through (13) yield a second order differential equation for $\widehat{w}$:

$$\left(f^2 - \omega^2\right) \frac{d^2 \widehat{w}}{dz^2} = -i2f\Lambda l \frac{d\widehat{w}}{dz} + \left(N^2 - \omega^2\right) l^2 \widehat{w}. \tag{14}$$

Ignoring any complications from boundary conditions, we find a simple solution

$$\widehat{w} = Ae^{imz} \tag{15}$$

which requires

$$\omega^2 = \frac{f^2 m^2 + 2f\Lambda l m + l^2 N^2}{m^2 + l^2}. \tag{16}$$

If we define

$$\beta \equiv \frac{l}{m} \tag{17}$$

(16) can be written as

$$\omega^2 = \frac{f^2 + 2f\Lambda\beta + \beta^2 N^2}{1 + \beta^2}. \tag{18}$$

Note the important point: ω^2 depends only on the ratio l/m and not on the wavenumber itself, small wavenumber and large wavenumber modes will have the same growth rate provided that β is the same. We now ask in what circumstance $\omega^2 < 0$, and hence ω would be imaginary and the mode would be unstable. The minimum of the numerator in (18) is found by

$$\frac{d}{d\beta}\left(f^2 + 2f\Lambda\beta + \beta^2 N^2\right) = 2f\Lambda + 2\beta N^2 = 0 \tag{19}$$

and occurs at

$$\beta = \frac{-\Lambda f}{N^2} \tag{20}$$

where the value of the numerator is

$$f^2 + 2f\Lambda\beta + \beta^2 N^2 = f^2\left(1 - \frac{\Lambda^2}{N^2}\right). \tag{21}$$

The numerator is negative if the Richardson number $\frac{N^2}{\Lambda^2} < 1$. In Figure 1 we plot ω^2 as a function of β for the meteorologically relevant case of $f/N = 0.01$. We assume $\Lambda/N = 2$, which is a bit stronger than observed in nature. (In fact, after all this mathematical fuss, we will find that dry symmetric instability would be rare in the atmosphere).

The true minimum in Fig. 1 does not occur as given by (20). Instead the minimum is where

$$\frac{d\omega^2}{d\beta} = \frac{2f\Lambda\beta + 2N^2\beta}{1+\beta^2} - \frac{2\beta}{1+\beta^2}\omega^2 = 0 \tag{22}$$

or

$$\beta = \frac{-\Lambda f}{N^2 - \omega^2} \tag{23}$$

Substitution of (23) into (18) yields

$$\left(N^2 - \omega^2\right)\left(\omega^2 - f^2\right) + f^2\Lambda^2 = 0. \tag{24}$$

Thus (23) can also be written as

$$\beta = \frac{\left(\omega^2 - f^2\right)}{f\Lambda}. \tag{25}$$

We next discuss what these modes look like and how the instability "works". From (13):

$$\widehat{v} = -\frac{1}{\beta}\widehat{w}. \tag{26}$$

From (26) and (10):

$$\widehat{u} = -i\frac{(f+\beta\Lambda)}{\omega\beta}\widehat{w}. \tag{27}$$

From (27) and (9)

$$\widehat{b} = -i\frac{\left(f\Lambda + N^2\beta\right)}{\omega\beta}\widehat{w}. \tag{28}$$

From the vertical momentum equation (12) and (28)

$$\widehat{\varphi} = -\frac{1}{\omega m}\left[\left(N^2 - \omega^2\right) + \frac{f\Lambda}{\beta}\right]\widehat{w}. \tag{29}$$

From the horizontal momentum equation (11) and (26) and (27):

$$\widehat{\varphi} = -\frac{1}{\omega l}\left[\frac{\omega^2 - f^2}{\beta} - f\Lambda\right]\widehat{w} \tag{30}$$

Using (18), (29) and (30) can be shown to be identical (of course). In an unstable mode, ω is imaginary and (27) indicates that u will be proportional to w. Thus instability consists of interleaving, slanted updrafts and downdrafts. Note the important fact that in the most unstable mode $\widehat{\phi} = 0$, in (29) or (30), by using (23) and (24). Because of this fact, the "physical explanation" for the most unstable mode is relatively simple.

Before we turn to that explanation, we reformulate the zonal momentum equation in terms of a conserved variable M. Equation (3) is the linearized version of

$$\frac{Du}{Dt} - fv' = 0. \tag{31}$$

The above equation can also be written as

$$\frac{DM}{Dt} = 0, \tag{32}$$

where

$$M \equiv f\left(u - fy\right). \tag{33}$$

Thus (3) can also be written as

$$\frac{\partial M'}{\partial t} = -w'\frac{\partial M_o}{\partial z} - v'\frac{\partial M_o}{\partial y}. \tag{34}$$

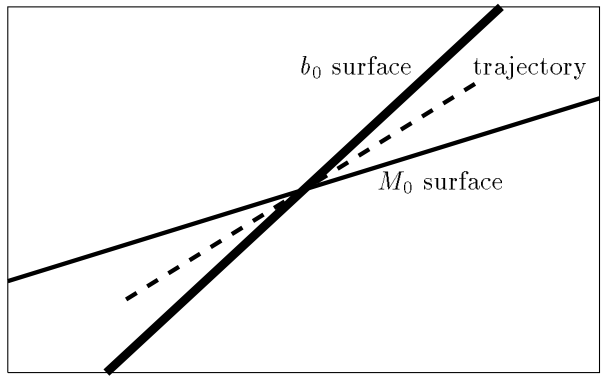

Figure 2: Schematic of symmetric instability in the (y, z) plane. Both b_0 and m_0 increase upward.

Likewise, the b' equation can be written as

$$\frac{\partial b'}{\partial t} = -w' \frac{\partial b_o}{\partial z} - v' \frac{\partial b_o}{\partial y}. \tag{35}$$

Now

$$dM_o = \frac{\partial M_o}{\partial z} dz + \frac{\partial M_o}{\partial y} dy. \tag{36}$$

Surfaces upon which M_0 is a constant are

$$\left(\frac{\partial z}{\partial y}\right)_{M_o} = \frac{-\partial M_o/\partial y}{\partial M_o/\partial z} = \frac{f}{\Lambda}. \tag{37}$$

Similarly,

$$\left(\frac{\partial z}{\partial y}\right)_{b_o} = \frac{-\partial b_o/\partial y}{\partial b_o/\partial z} = \frac{\Lambda f}{N^2}. \tag{38}$$

The condition for instability is $\Lambda^2 > N^2$, which implies

$$\left(\frac{\partial z}{\partial y}\right)_{b_o} > \left(\frac{\partial z}{\partial y}\right)_{M_o} \tag{39}$$

An unstable situation is shown in Figure 2, which shows a representative b_0 and M_0 surface. Also shown is a "trajectory" of a parcel in the instability. Actually, we show a line parallel to the stream line, the movement could be in either direction, depending on whether it was in the "updraft" or "downdraft". The orientation of the trajectory is calculated as follows: Along lines of constant phase, which are also lines of constant stream function and hence streamlines,

$$l dy + m dz = 0 \tag{40}$$

or

$$\frac{dz}{dy} = \frac{-l}{m} = -\beta. \tag{41}$$

From (27) we can write:

$$\frac{dz}{dy} = \frac{(f^2 - \omega^2)}{f^2} \frac{f}{\Lambda}. \tag{42}$$

In an instability, $\omega^2 < 0$, so (42) compared with (37) says that

$$\left(\frac{\partial z}{\partial y}\right)_{M_o} < \frac{dz}{dy}. \tag{43}$$

Using (23) in (41) :

$$\frac{dz}{dy} = \frac{\Lambda f}{N^2 - \omega^2}. \tag{44}$$

Comparing with (38),

$$\frac{dz}{dy} < \left(\frac{\partial z}{\partial y}\right)_{b_o}. \tag{45}$$

Thus (43) and (45) taken together are

$$\left(\frac{\partial z}{\partial y}\right)_{M_o} < \frac{dz}{dy} < \left(\frac{\partial z}{\partial y}\right)_{b_o}. \tag{46}$$

Now, in the most unstable mode, ϕ'=0, so

$$\frac{\partial\, v'}{\partial\, t} = -M'. \tag{47}$$

In the upwards part of the trajectory, where $w' > 0$ and $v' > 0$ are positive, conservation of M will make $M' < 0$ making the parcel "want to keep on going" with $\partial v'/\partial t > 0$. Similarly, with

$$\frac{\partial\, w'}{\partial\, t} = b' \tag{48}$$

conservation of b will make $b' > 0$ and the parcel will "want to keep on going" with $\partial w'/\partial t > 0$. The "'want" is not becuase of any concious thought of the paracels, but is becase the parcels are in exactly the unmodified pressure field of the basic state. An analysis of the energetics of symmetric instability also provides an "explanation" of how it works. We can derive the following from (2)-(6):

$$\frac{\partial}{\partial\, t}\left[\frac{1}{2}\left\langle\overline{u'^2}\right\rangle + \frac{1}{2}\left\langle\overline{v'^2}\right\rangle + \frac{1}{2}\left\langle\overline{w'^2}\right\rangle\right] = -\Lambda\left\langle\overline{u'w'}\right\rangle + \left\langle\overline{w'b'}\right\rangle \tag{49}$$

and

$$\frac{\partial}{\partial\, t}\left[\frac{1}{2N^2}\left\langle\overline{b'^2}\right\rangle\right] = \frac{f\Lambda}{N^2}\left\langle\overline{v'b'}\right\rangle - \left\langle\overline{w'b'}\right\rangle. \tag{50}$$

Adding (49) and (50)

$$\frac{\partial}{\partial\, t}\left[\frac{1}{2}\left\langle\overline{u'^2}\right\rangle + \frac{1}{2}\left\langle\overline{v'^2}\right\rangle + \frac{1}{2}\left\langle\overline{w'^2}\right\rangle + \frac{1}{2N^2}\left\langle\overline{b'^2}\right\rangle\right] = -\Lambda\left\langle\overline{u'w'}\right\rangle + \frac{f\Lambda}{N^2}\left\langle\overline{v'b'}\right\rangle. \tag{51}$$

By inspection of the trajectories, and of formal analysis of (26)-(30), both terms on the right of (51) are positive, thus increasing the kinetic energy of the "disturbance".

2 Linear stability analysis between two rigid boundaries

We next turn to a linear stability analysis with rigid boundaries at z=0 and z=H, so that

$$\widehat{w}(0) = 0 \tag{52}$$

and

$$\widehat{w}(H) = 0. \tag{53}$$

A single mode of form (15) cannot satisfy (52)-(53). However if we superpose two modes of the same frequency and same wavenumber l, but differing m and b, it turns out to be possible to satisfy (52)-(53). For this purpose, the frequency will have to be above the minimum in Fig. 1. We use (16) to calculate m:

$$m^2\left(f^2 - \omega^2\right) + 2f\Lambda l m + \left(N^2 - \omega^2\right) l^2 = 0 \tag{54}$$

which yields two positive roots:

$$m = \frac{-f\Lambda l}{f^2 - \omega^2} \pm \frac{1}{(f^2 - \omega^2)} \sqrt{f^2 \Lambda^2 l^2 + (\omega^2 - f^2)(N^2 - \omega^2) l^2}. \tag{55}$$

So with

$$\widehat{w} = A e^{i m_1 z} + B e^{i m_2 z}, \tag{56}$$

conditions (52)-(53) require

$$A + B = 0, \tag{57}$$

$$A e^{i m_1 H} + B e^{i m_2 H} = 0. \tag{58}$$

The above combine to give

$$B e^{i m_1 H} = B e^{i m_2 H} \tag{59}$$

which occurs for

$$(m_2 - m_1) H = 2 n \pi. \tag{60}$$

where n is an integer. Using (55) in (60), we see we must choose ω^2 to satisfy

$$\frac{2H}{(f^2 - \omega^2)} \sqrt{f^2 \Lambda^2 l^2 + (\omega^2 - f^2)(N^2 - \omega^2) l^2} = 2 n \pi \tag{61}$$

In order to have no internal levels $0 < z < H$ where $\omega = 0$ (an internal "pseudo-rigid boundary"), we should take $n = 1$ or $n = -1$, and ω^2 to satisfy

$$\frac{1}{(f^2 - \omega^2)} \sqrt{f^2 \Lambda^2 + (\omega^2 - f^2)(N^2 - \omega^2)} = \frac{n\pi}{Hl} \equiv \frac{1}{\alpha} \tag{62}$$

or

$$\omega^2 = f^2 + \frac{(N^2 - f^2)\alpha^2}{2(1 + \alpha^2)} \pm \frac{1}{2(1 + \alpha^2)} \sqrt{(N^2 - f^2)^2 \alpha^4 + 4 f^2 \Lambda^2 \alpha^2 (1 + \alpha^2)}. \tag{63}$$

It turns out (though it will not be shown here) that the minimum of (63) occurs as $\alpha \to \infty$ and $\omega^2 < 0$ if $\Lambda^2 > N^2$. This is entirely expected by looking at Fig. 1, and seeing that the most unstable mode occurs when we can use a pair of modes with β very close to the minimum given by (23). We therefore have found that the most unstable mode between rigid boundaries will occur as $l \to \infty$ and $m \to \infty$. This property of the instability is quite distinct from unbounded symmetric instability, in which ω^2 depends only on b. In a real viscous fluid, viscous damping becomes very effective at large wavenumber, as in convective instability. Stability analysis of that situation, which is more complicated, would show that the maximum growth rate would occur short of the limit of infinite wavenumber. Even in an ideal fluid, a plot of (63) shows that ω^2 becomes quite flat for $\alpha > 3$, and very little viscosity, numerical of otherwise, is required to offset the "advantage" of $\alpha \to \infty$. As a bonus, equation (62) provides an easy way to calculate the frequency of the most unstable mode (which is exactly (24)), which occurs as $\alpha \to \infty$:

$$(N^2 - \omega^2)(\omega^2 - f^2) + f^2 \Lambda^2 = 0. \tag{64}$$

In the most unstable mode,

$$m_1 \simeq m_2 \tag{65}$$

and the slope of either of the individual modes found within the most unstable mode is very close to most unstable mode discussed in section 1, that is with

$$\frac{l}{m} = \beta = \frac{(\omega^2 - f^2)}{f\Lambda}. \tag{66}$$