An Unstable Parallel Flow

Brian Fiedler

March 2, 1998

Dynamical instabilities are one of the main causes of what we call "weather". The most notorious of these are convective instability and baroclinic instability. We now embark on a series of studies of the stability of wind shear in a stably stratified atmosphere ($N^2 \geq 0$). We begin here with one of the simplest possible configurations and consider $N^2 = 0$ and $f = 0$. We will find that wind shear is destabilizing. Admittedly, this configuration is not directly relevant in meteorology, but it illustrates some important processes that will be important later.

The inviscid, incompressible two-dimensional equations in the absence of buoyancy or a Coriolis force are:

$$\frac{Du}{Dt} = -\frac{\partial \varphi'}{\partial x}, \tag{1}$$

$$\frac{Dw}{Dt} = -\frac{\partial \varphi'}{\partial z}, \tag{2}$$

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0. \tag{3}$$

Equations (1) - (3) combine to give a vorticity equation

$$\frac{D\omega}{Dt} = 0, \tag{4}$$

where

$$\omega \equiv \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}. \tag{5}$$

This definition is in deference to students of meteorology, who would probably favor positive vorticity being seen as cyclonic (anti-clockwise) in a plot of the $x - z$ plane.

Next we define the define the velocity as the sum of a basic shear flow $u_o(z)$ and a departure from this flow:

$$u = u_o(z) + u' \tag{6}$$

and

$$w = w'. \tag{7}$$

Equation (4) becomes

$$\left(\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x} + u'\frac{\partial}{\partial x} + w'\frac{\partial}{\partial z}\right)\left(\frac{\partial w'}{\partial x} - \frac{d\,u_o}{d\,z} - \frac{\partial u'}{\partial z}\right) = 0 \tag{8}$$

Linearizing (8) and using a stream function to define w' and u',

$$w' = \frac{\partial \psi}{\partial x} \tag{9}$$

$$u' = -\frac{\partial \psi}{\partial z} \tag{10}$$

gives

$$\left(\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x}\right)\nabla^2\psi - \frac{\partial\,\psi}{\partial\,x}\frac{d^2\,u_o}{d\,z^2} = 0. \tag{11}$$

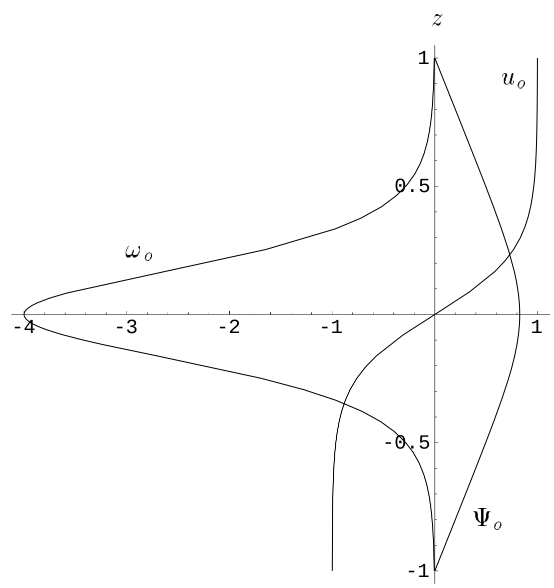

Figure 1: The basic shear flow to be analyzed (it's unstable!).

Proposing separable solutions of form

$$\psi \equiv \widehat{\psi}(z)e^{ik(x-ct)}, \tag{12}$$

we arrive at the *Rayleigh equation*

$$\left(D^2 - k^2\right)\widehat{\psi} - \frac{1}{u_o - c}\frac{d^2\, u_o}{d\, z^2}\widehat{\psi} = 0 \tag{13}$$

where

$$D \equiv \frac{d}{dz}. \tag{14}$$

In the analysis that follows we assume that the variables have been scaled so that they are dimensionless. We consider a particular shear profile of Figure 1:

$$u_o(z) = \tanh(\alpha z), \tag{15}$$

with $\alpha = 4$ and rigid boundaries at $z = -1$ and $z = 1$. Thus, the initial or basic-state vorticity is concentrated in the middle of the layer:

$$-\frac{d\, u_o}{d\, z} = -\alpha \mathrm{sech}^2(\alpha z) \tag{16}$$

With $w = 0$ at the top and bottom boundaries, $\widehat{\psi} = 0$ at those boundaries. We use a numerical method (a shooting scheme) to solve (13) and find eigenvalues of c (for a given k) that allow for nontrivial ($\widehat{\psi} \neq 0$) solutions that satisfy the boundary conditions. We are able to find unstable solutions, solutions with $c_i > 0$. The complex eigenfunction $\widehat{\psi}(z)$ for $k = 2$ is shown in Figure 2. $\psi(x, z, t)$ is shown in Figure 3 for this unstable mode. We choose to analyze $k = 2$, because $k = 2$ gives a growth rate kc_i near the maximum value as seen in Figure 4.

Next we consider the nonlinear evolution of the instability at $k = 2$ using the numerical model **EZM**. We solve (1)-(3) in a domain of height 2 and width $2\pi/k$. The integration is initialized with

$$u(t=0) \equiv u_o + \varepsilon \cos(kx) \sin\left(\frac{\pi}{2}z\right) \tag{17}$$

and

$$w(t=0) \equiv -\varepsilon \frac{2k}{\pi} \sin(kx) \cos\left(\frac{\pi}{2}z\right) \tag{18}$$

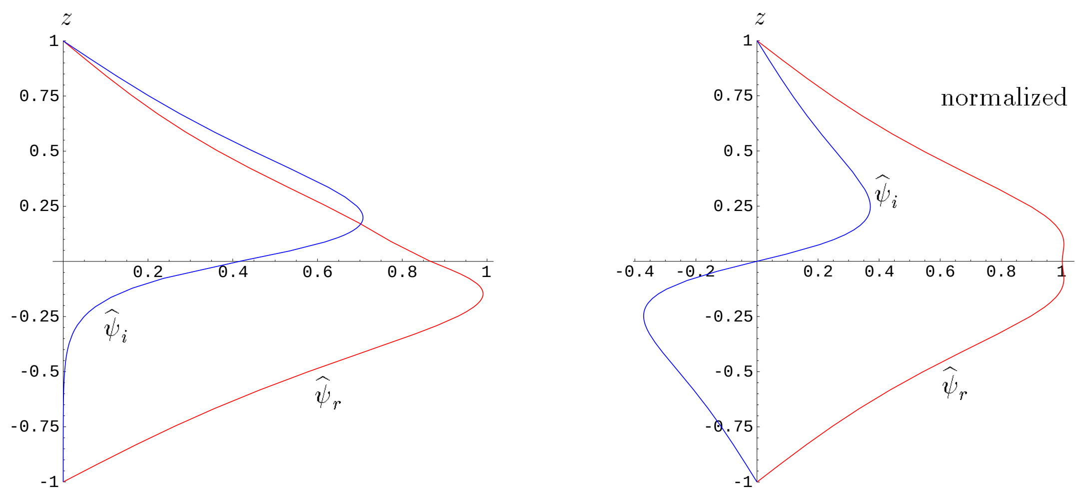

Figure 2: The complex eigenfunction $\widehat{\psi}(z)$ for $k = 2$. The plot on the right is normalized so that $\Im[\widehat{\psi}(0)] = 0$.

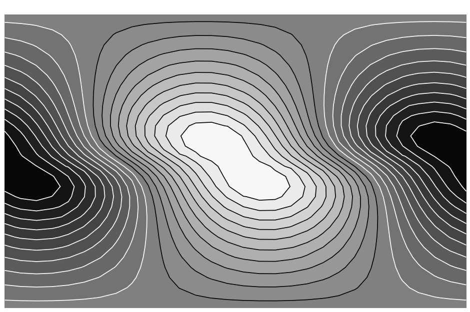

Figure 3: ψ in the $x - z$ plane for the unstable mode. Whiter is more positive.

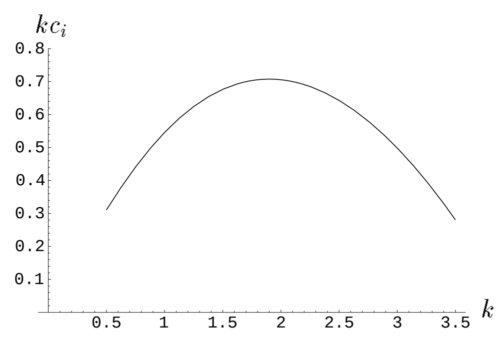

Figure 4: The growth rate kc_i as a function of k.

where

$$\varepsilon = 0.0001. \tag{19}$$

In order to visualize the flow we define a total stream function Ψ:

$$w \equiv \frac{\partial\, \Psi}{\partial\, x} \tag{20}$$

$$u \equiv -\frac{\partial\, \Psi}{\partial\, z}. \tag{21}$$

The growth rate during the time of exponential growth $(2 < t < 12)$ (Figure 5) is diagnosed to be $kc_i = 0.698$, while in the stability analysis it was found to be $kc_i = 0.706$.

We now define

$$u = \overline{u} + u' \tag{22}$$

and

$$w = \overline{w} + w' \tag{23}$$

where the overbar denotes the horizontal average. Note that the prime ($'$) now denotes a departure from the horizontal average, rather than departure from the basic state. This notation can be confusing; the reader will have to remember how ($'$) is being used in a particular equation.

Notice in Figure 6 the tilt of the streamfunction fluctuation Ψ' tilts into the shear during the exponential growth phase at $t = 5$, $t = 10$ and $t = 13$. This tilt is the key to the instability mechanism, which we now demonstrate. It can be easily shown from (1)-(3) that the total kinetic energy is conserved, or that $\left\langle \overline{u^2 + w^2} \right\rangle$ is conserved where $\langle\rangle$ denotes the vertical average. Also, it is easy to show that $\overline{w} = 0$ so

$$\frac{\partial}{\partial t}\left\langle \overline{u'^2 + w'^2} + \overline{u}^2 \right\rangle = 0. \tag{24}$$

The horizontal momentum equation can be written as

$$\frac{\partial\left(\overline{u} + u'\right)}{\partial t} + \frac{\partial}{\partial x}\left(\overline{u} + u'\right)\left(\overline{u} + u'\right) + \frac{\partial}{\partial z}\left(\overline{w} + w'\right)\left(\overline{u} + u'\right) = -\frac{\partial(\overline{\varphi} + \varphi')}{\partial x} \tag{25}$$

Now if we apply the horizontal average to (25) we get

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial z}\overline{w'u'} = 0 \tag{26}$$

Multiplying (26) by $\overline{u}$, taking the vertical average, and itegrating by parts gives:

$$\frac{\partial}{\partial t}\left\langle \frac{\overline{u}^2}{2} \right\rangle = \left\langle \overline{w'u'}\frac{\partial \overline{u}}{\partial z} \right\rangle. \tag{27}$$

Now using (24) we have

$$\frac{\partial}{\partial t}\left\langle \frac{\overline{u'^2 + w'^2}}{2} \right\rangle = -\left\langle \overline{w'u'}\frac{\partial \overline{u}}{\partial z} \right\rangle \tag{28}$$

In the example here, initially $\frac{\partial \overline{u}}{\partial z} > 0$. The energy of the perturbation will increase only if u' and w' are negatively correlated, or if streamlines tilt up and towards the left.

Kundu presents more details of the linear instability of the system (1)-(3). Some important points are that not just any shear profile is unstable. Rayleigh's and Fjortoft's theorem on page 392, taken together, state that a necessary condition for instability is that the basic flow $u_o(z)$ must have a local maximum in the magnitude of vorticity, or $|\frac{du_o}{dz}|$, away from the boundaries. One consequence of this condition is that a linear profile of constant $\frac{du_o}{dz}$ would always be stable. This may seems like a peculiar result. If one considered qualitative arguments based on energetics, it would seem that tilted circulations, as in the example shown here, could also extract energy and drive an instability in a linear shear - but this is not the case. Most

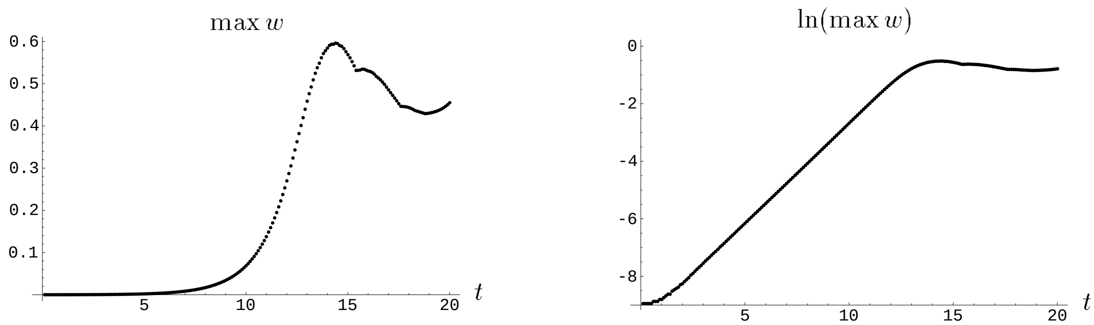

Figure 5: Maximum value of w as a function of time in the numerical simulation.

peculiar, indeed! Kundu also presents the more general system with a basic state of $N^2 > 0$ and advection of buoyancy. He presents the necessary condition for instability that

$$R \equiv \frac{N^2}{\left(\frac{du_o}{dz}\right)^2} < \frac{1}{4} \tag{29}$$

somewhere in the flow. The dimensionless ratio R is usually called the Richardson number R. This result seems intuitively proper: large N^2 will prevent fluid from overturning.

Kundu's study of stratified shear flows, in the absense of a Coriolis force, is relatively easy, but of less meteorological relevance, compared with what we will attempt next. In subsequent treatises we will study the stability of stratified shear flows in thermal wind balance, that is where the vertical shear is associated with a horizontal buoyancy gradient:

$$\frac{\partial b_o}{\partial y} = -f\frac{\partial u_o}{\partial z}. \tag{30}$$

We will find a symmetric instability that happens with $R < 1$, and a baroclinic instability that occurs for any $0 < R < \infty$.

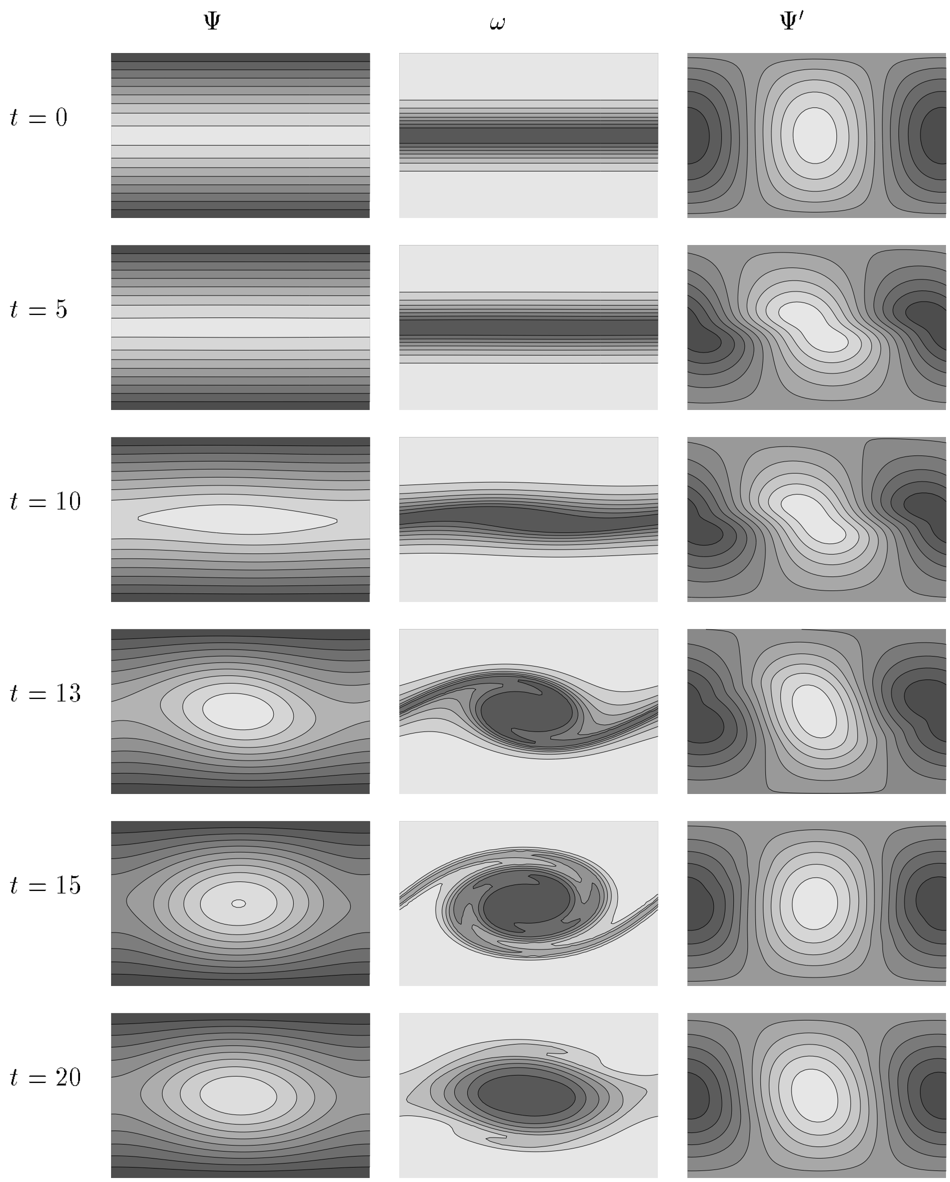

Figure 6: Integration (forecast, simulation) of the unstable flow with a numerical model. Whiter is more positive. Note that $D\omega/Dt = 0$ is not satisfied exactly, due to the approximate numerical nature of the model.