The Shallow Water Canon

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Shallow water theory is used here to "illustrate" two phenomena that also occur in a stably stratified atmosphere: the Rossby wave and geostrophic adjustment. We use the word "illustrate" rather than "model" because the theory has little quantitative predictive ability without explicitly considering the equations for a continuously stratified fluid, like the atmosphere. The phenomena predicted here will differ from those in the atmosphere by both either the time scale, the spatial scale, or both. Shallow water theory is used here because the mathematical analysis is easier. The student will have to accept as an article of faith that similar phenomena occur in atmospheric models and the real atmosphere. The linearized shallow-water system on either an f-plane or β-plane is:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{2}$$

$$\frac{\partial \eta}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{3}$$

1 Geostrophic adjustment

Geostrophic adjustment concerns how the atmosphere reaches and maintains geostrophic balance, defined as a state where the horizontal pressure gradient force closely balances the Coriolis force. We will demonstrate with shallow water theory that for disturbances with large spatial scales, the wind will adjust, thereby allowing the Coriolis force to adapt to the pressure gradient force. For small spatial scales the height field will adjust, thereby allowing the pressure gradient force to adapt to the Coriolis force. This demonstration is made using the f-plane approximation, $f = f_o = constant$. The separating length scale between "large" and "small" is the Rossby radius of deformation L_R:

$$L_R = \sqrt{\frac{gH}{f_o^2}}. \tag{4}$$

If v is eliminated between (1) and (2) then

$$\frac{\partial^2 u}{\partial t^2} + f_o^2 u = -g\frac{\partial^2 \eta}{\partial x \partial t} - gf_o\frac{\partial \eta}{\partial y}. \tag{5}$$

If u is eliminated between (1) and (2) then

$$\frac{\partial^2 v}{\partial t^2} + f_o^2 v = -g\frac{\partial^2 \eta}{\partial y \partial t} + gf_o\frac{\partial \eta}{\partial x}. \tag{6}$$

If u and v are now eliminated from (3) we have

$$\left(\frac{\partial^2}{\partial t^2} + f_o^2\right)\frac{\partial \eta}{\partial t} - gH\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)\frac{\partial \eta}{\partial t} = 0. \tag{7}$$

Consider the $\partial/\partial y = 0$. Eqn. (7) admits a Poincaré wave solution

$$\eta_w\,(x,t) = A\cos(kx)\cos(\omega t), \tag{8}$$

where

$$\omega^2 = f_o^2 + gHk^2. \tag{9}$$

The corresponding velocity field for the wave is obtained from (5) and (6):

$$u_w\,(x,t) = A\frac{\omega}{Hk}\sin(kx)\sin(\omega t) \tag{10}$$

and

$$v_w\,(x,t) = A\frac{f_o}{Hk}\sin(kx)\cos(\omega t). \tag{11}$$

The governing equations (1)-(3) also admit a steady geostrophic solution:

$$\eta_g\,(x) = B\cos(kx), \tag{12}$$

$$u_g\,(x) = 0 \tag{13}$$

and

$$v_g\,(x) = -B\frac{gk}{f_o}\sin(kx) \tag{14}$$

Since the governing equations are linear, the geostrophic solution and wave solution can be superposed to give:

$$\eta\,(x,t) = A\cos(kx)\cos(\omega t) + B\cos(kx), \tag{15}$$

$$u\,(x,t) = A\frac{\omega}{Hk}\sin(kx)\sin(\omega t) \tag{16}$$

and

$$v\,(x,t) = A\frac{f_o}{Hk}\sin(kx)\cos(\omega t) - B\frac{gk}{f_o}\sin(kx). \tag{17}$$

Consider $\eta(x,0) = \varepsilon\cos(kx)$, $u(x,0) = 0$, and $v(x,0) = -\delta\frac{gk}{f_0}\sin(kx)$. The height field that would be in geostrophic equilibrium with the initial $v(x,0)$ is $\eta(x) = \delta\cos(kx)$. Below we show that if $\delta \neq \varepsilon$, a wave is generated, and the amplitude of the steady part, or B, falls in between δ and ε. From (16),

$$A + B = \varepsilon \tag{18}$$

and from (17)

$$A - k^2L_R^2B = -k^2L_R^2\delta \tag{19}$$

The solution of (18) and (19) is

$$A = (\varepsilon - \delta)\,\frac{k^2L_R^2}{1+k^2L_R^2} \tag{20}$$

and

$$B = \frac{\varepsilon + \delta k^2L_R^2}{1+k^2L_R^2} \tag{21}$$

Notice that the amplitude of the geostrophic response, B, is a weighted average of the amplitudes ε and δ . If $k^2L_R^2 << 1$ then ε dominates B. If $k^2L_R^2 >> 1$ then δ dominates B. We sometimes say that : "In large scale geostrophic adjustment, the wind field adjusts to the height field. In small scale geostrophic adjustment, the height field adjusts to the wind field." This fact helps explain the dominance of the height field in large-scale meteorological models and, in particular, quasi-geostrophic models and theories. The above points can be emphasized by considering the energetics of (1)-(3). If we multiply (1) by Hu, (2) by Hv, and (3) by $g\eta$, and add all the equations, we get:

$$\frac{\partial}{\partial t}\left(H\frac{u^2}{2} + H\frac{v^2}{2} + g\frac{\eta^2}{2}\right) = -\frac{\partial}{\partial x}\,(guH\eta) - \frac{\partial}{\partial y}\,(\;gvH\eta) \tag{22}$$

If we consider a system with either rigid or periodic lateral boundaries, then

$$\frac{\partial}{\partial t}\frac{1}{L_1 L_2}\int_0^{L_2}\int_0^{L_1}\left(H\frac{u^2}{2}+H\frac{v^2}{2}+g\frac{\eta^2}{2}\right)dxdy = 0 \tag{23}$$

Consider a geostrophic state with $u = 0$ ($A = 0$),

$$\frac{1}{L_1 L_2}\int_0^{L_2}\int_0^{L_1}\left(H\frac{v^2}{2}+g\frac{\eta^2}{2}\right)dxdy = \frac{B^2 g}{4}\left(k^2 L_R^2 + 1\right) \tag{24}$$

The term$k^2 L_R^2$ is from the kinetic energy, and it is small compared to the potential energy if $k^2 L_R^2 << 1$. At large scales (small k), the pressure gradient is relatively small for a given displacement. The velocity field in large scales can adjust without requiring much transfer of the potential energy, and thus the height field can be relatively unaffected by any geostrophic adjustment. The opposite is true if $k^2 L_R^2 >> 1$. In that case, with large k, only relatively small changes in η are needed to change the pressure field to match the Coriolis force and little energy needs to be transferred from kinetic energy for that adjustment.

In the above simple presentation, the gravity wave neither dissipates nor propagates away. It would not be formally correct to say the system "adjusts" to a geostrophic state; the solution is periodic in time and the initial state "returns" again and again. More elaborate adjustment scenarios can be concocted by considering an isolated disturbance, rather than a periodic disturbance. In in Fig. 1, we consider an initial "tophat" function in η:

$$\eta(x,0) = 1, \quad |x| < L. \tag{25}$$

and $v(x,0) = 0$. The Fourier transform is:

$$\begin{aligned}\hat{\eta}(k,0) &= \int_{-L}^{L}\cos kx dx \\ &= \frac{2\sin(kL)}{k}\end{aligned} \tag{26}$$

The Fourier transform of the geostrophic part is:

$$\hat{\eta}_g(k,0) = \frac{1}{1+k^2 L_R^2}\frac{2\sin(kL)}{k} \tag{27}$$

The inverse Fourier transform gives the steady geostrophic state left behind after all the waves propagate way:

$$\begin{aligned}\eta_g(x,0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\eta}_g(k,0)\cos(kx)dk \\ &= \frac{1}{2}\mathrm{sign}(L-x)\left[1-\exp\left(-\sqrt{\frac{(L-x)^2}{L_R^2}}\right)\right]+\frac{1}{2}\mathrm{sign}(L+x)\left[1-\exp\left(-\sqrt{\frac{(L+x)^2}{L_R^2}}\right)\right]\end{aligned} \tag{28}$$

If $L_R << L$ then the initial height field is left unaltered. If $L_R >> L$, then the final height field is much broader and flatter than the initial field; it broadens to the scale of order L_R.

2 Rossby Waves

Rossby waves in their pure form are not easily seen in atmospheric data, and are somewhat over-rated in meteorology. Their popularity is a result of providing one of the most glorious demonstrations of the efficacy of the quasi-geostrophic approximation, an approximation that filters out high-frequency gravity waves from the mathematical analysis, rendering the analysis much more tractable. Here we make the $\beta-$plane approximation and assume

$$f = f_o + \beta y \tag{29}$$

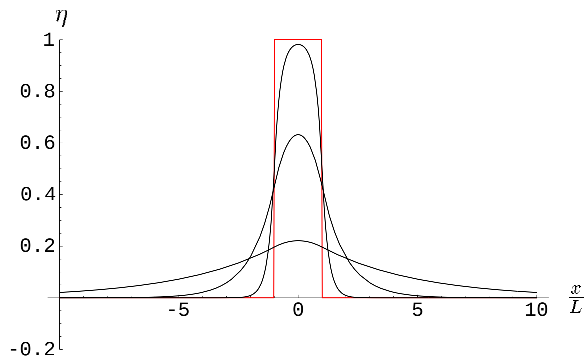

Figure 1: A plot of (28) for $\frac{L_R}{L}$= 0, 0.25, 1 and 4 (in descending central amplitude).

where b and f_o are constants. The variation of the Coriolis parameter with latitude will be assumed to be "slight", so

$$|\beta y| << f_o \tag{30}$$

within the bounds for y. The $\beta-$plane is assumed to be infinite in the horizontal. If we propose a steady geostrophic solution

$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} \tag{31}$$

and

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y}, \tag{32}$$

we encounter a contradiction with our steady assumption - the horizontal flow is divergent (Fig. 3):

$$\frac{1}{H}\frac{\partial \eta}{\partial t} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{g}{f^2}\beta\frac{\partial \eta}{\partial x} \cong \frac{g}{f_o^2}\beta\frac{\partial \eta}{\partial x} \tag{33}$$

Equation (33) suggests a wave solution of $cos(kx - \omega t)$ with dispersion relation

$$\omega = \frac{-k\beta}{\frac{f_o^2}{gH}} \tag{34}$$

The wave solution violates the steady geostrophic assumption (31) that was used to derive (33), however is does hint that a nearly geostrophic solution might move westward. The vorticity equation derived from (1) and (2) is:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{f}{H}\frac{\partial \eta}{\partial t} + \frac{df}{dy}v = 0 \tag{35}$$

Suppose we impose $\eta \to 0$, implying $w \to 0$ as well. The horizontal flow must then be nondivergent, so we define a streamfunction

$$v = \frac{\partial \psi}{\partial x} \tag{36}$$

and

$$u = -\frac{\partial \psi}{\partial y}. \tag{37}$$

So (35) is:

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0 \tag{38}$$

For waves of form $e^{i(kx+my-\omega t)}$ we get a dispersion relation

$$\omega = \frac{-\beta k}{k^2 + m^2} \tag{39}$$

Thus, for example, a "high" in the streamfunction, and its associated negative vorticity and anti-cyclonic circulation, will tend to move to the west due to the advection up the gradient of planetary vorticity in the southerly flow on the west of the high. A similar statement can be made about the westward movement of the low.

If we take away the rigid lid in the above scenario, the anti-cyclonic vorticity and associated high pressure would have to be accompanied by a high in η. The movement of the high westward would require the convergence indicated by the first scenario. However, the convergence that builds the high also stretches planetary vorticity and tends to create cyclonic vorticity and counteracts the advection term. This suggests that westward movement would be less rapid than that implied by (39). Indeed, the following analysis shows exactly that. Equations (1) and (2) imply that in low frequency waves, or $\omega << f_o$, either u or v (or both) should be geostrophic. We next try a quasi-geostrophic solution in which we expect the wave motion to be of low enough frequency that we can express the vorticity in (35) by using the geostrophic wind (31):

$$\frac{\partial}{\partial t}\frac{g}{f_o}\nabla^2\eta - \frac{f_o}{H}\frac{\partial\eta}{\partial t} + \frac{\beta g}{f_o}\frac{\partial\eta}{\partial x} = 0 \tag{40}$$

In (40) we have also made an approximation analogous to the Boussinesq approximation; we have ignored variations in f except where it generates vorticity. The dispersion relation for Rossby waves as derived from (40) is

$$\omega = \frac{-k\beta}{k^2 + m^2 + \frac{f_o^2}{gH}} \tag{41}$$

Our expectations to have smaller frequency than that in the approximate model (38) is confirmed – the frequency appears to be a hybrid of (39) and (34). Note that (40) can be obtained quite easily by linearizing the potential vorticity equation

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0 \tag{42}$$

and then using the geostrophic wind in the relative vorticity ζ. How confident are we that our quasi-geostrophic model is correct? We can solve (1)- (3) in all their glory. First we eliminate η in (1) and (2):

$$\frac{\partial^2 u}{\partial t^2} - f\frac{\partial v}{\partial t} = gH\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{43}$$

$$\frac{\partial^2 v}{\partial t^2} + f\frac{\partial u}{\partial t} = gH\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{44}$$

or

$$\left(\frac{\partial^2}{\partial t^2} - gH\frac{\partial^2}{\partial x^2}\right)u = \left(gH\frac{\partial^2}{\partial y\partial x} + f\frac{\partial}{\partial t}\right)v \tag{45}$$

$$\left(\frac{\partial^2}{\partial t^2} - gH\frac{\partial^2}{\partial y^2}\right)v = \left(gH\frac{\partial^2}{\partial y\partial x} - f\frac{\partial}{\partial t}\right)u \tag{46}$$

Next we eliminate u. We first "operate" on (45):

$$\left(gH\frac{\partial^2}{\partial y\partial x} - f\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial t^2} - gH\frac{\partial^2}{\partial x^2}\right)u = \left(gH\frac{\partial^2}{\partial y\partial x} - f\frac{\partial}{\partial t}\right)\left(gH\frac{\partial^2}{\partial y\partial x} + f\frac{\partial}{\partial t}\right)v \tag{47}$$

We can interchange the operators on the left. (But not on the right, can you see why?)

$$\left(\frac{\partial^2}{\partial t^2} - gH\frac{\partial^2}{\partial x^2}\right)\left(gH\frac{\partial^2}{\partial y\partial x} - f\frac{\partial}{\partial t}\right)u = \left(gH\frac{\partial^2}{\partial y\partial x} - f\frac{\partial}{\partial t}\right)\left(gH\frac{\partial^2}{\partial y\partial x} + f\frac{\partial}{\partial t}\right)v \tag{48}$$

Next we make a substitution from (46) to eliminate u:

$$\left(\frac{\partial^2}{\partial t^2} - gH\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - gH\frac{\partial^2}{\partial y^2}\right)v = \left(gH\frac{\partial^2}{\partial y\partial x} - f\frac{\partial}{\partial t}\right)\left(gH\frac{\partial^2}{\partial y\partial x} + f\frac{\partial}{\partial t}\right)v. \tag{49}$$

Proposing separable solutions of form

$$(u, v, w) = [\widehat{u}(y), \widehat{v}(y), \widehat{\eta}(y)]e^{i(kx-\omega t)}, \tag{50}$$

(49) becomes:

$$\left(-\omega^2+gHk^2\right)\left(-\omega^2-gHD^2\right)\widehat{v} = -\left(gHkD + f\omega\right)\left(gHkD - f\omega\right)\widehat{v}, \tag{51}$$

or

$$\omega\left[\omega^3 + gH(D^2-k^2)\omega - f^2\omega - gH\beta k\right]\widehat{v} = 0, \tag{52}$$

where

$$D \equiv \frac{d}{dy} \tag{53}$$

We can find approximate solutions to (52) if we replace $f(y)$ by f_o because of (30). We get a constant coefficient equation and find $\widehat{v}(y) = A\sin(mz)$ if

$$\omega\left[\omega^3 - gH(k^2+m^2)\omega - f_o^2\omega - gH\beta k\right] = 0 \tag{54}$$

The above equation has four roots for ω, one of which is close to our quasi-geostrophic solution (41) if we ignore the ω^3 term because $\omega << f_o$. There are also two high-frequency modes for gravity waves modified by rotation, given approximately by

$$\omega^2 = gH(k^2+m^2) + f_o^2 \tag{55}$$

The root $\omega = 0$ gives a steady solution, with $k = 0$ and $v = 0$; the disturbance is a steady modification of the zonal geostrophic wind. The ratio of some suitably defined frequency ω to the Coriolis parameter f_o is called the Rossby number:

$$R_o = \left|\frac{\omega}{f_o}\right| \tag{56}$$

In scale analysis, this "frequency" is sometimes calculated by U/L where U is the typical windspeed in a large-scale weather event, and L is the length scale of the event. As claimed on page 484 of Kundu, $R_o \approx 0.1$ in mid-latitudes of the Earth's atmosphere. As a result, there are many theories or models in which the wind is primarily diagnosed from the pressure field, as in (40).

But why deny ourselves the joy of solving (52) as a non-constant coefficient equation? We can write (52) as:

$$D^2\widehat{v} + \left(-k^2 - \frac{f^2}{gH} + \frac{\omega^2}{gH} - \frac{\beta k}{\omega}\right)\widehat{v} = 0. \tag{57}$$

We can also choose to write:

$$f = f_o\left(1 + \frac{y}{L_f}\right). \tag{58}$$

Now if we use the Rossby radius of deformation L_R from (4) to nondimensionalize y and the wavenumber k, and also nondimensionalize ω with f_o , we can write (57) as:

$$D^2\widehat{v} + \left(-k^2 - \left(1 + \frac{y}{L_f}\right)^2 + \omega^2 - \frac{k}{\omega}\frac{L_R}{L_f}\right)\widehat{v} = 0. \tag{59}$$

Enjoy this equation in **Mathematica**!