Thermal Convection

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1 The model equations: nondimensionalization

Here we add some rudimentary knowledge of convective instability to our stockpile of meteorological instabilities. We will need to tack on the simplest sort of diffusion term to the inviscid equations we derived. With a constant viscosity ν, and no Coriolis force, the momentum equations are:

$$\frac{Du}{Dt} = -\frac{\partial \varphi'}{\partial x} + \nu \nabla^2 u \tag{1}$$

and

$$\frac{Dw}{Dt} = -\frac{\partial \varphi'}{\partial z} + b' + \nu \ \nabla^2 w, \tag{2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

The buoyancy equation with a constant diffusivity κ will be

$$\frac{Db}{Dt} = \kappa \nabla^2 b. \tag{4}$$

Recalling the total buoyancy b can be written as:

$$b = b_0(z) + b'(x, y, z, t) \tag{5}$$

we have

$$\frac{Db'}{Dt} = -wN^2 + \kappa \nabla^2 b' + \kappa \frac{d^2 b_0}{d z^2}. \tag{6}$$

Here we will require that the last term in (6) is zero in the cases that we study. We will also use the Boussinesq limit,

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x}. \tag{7}$$

The model equations with appropriate boundary conditions (which will be discussed later) form a dimensionally homogeneous system of equations with 11 dimensional variables:

$$F(b', u, w, \varphi', N^2, H, \nu, \kappa, x, z, t) = 0 \tag{8}$$

where F is some horrendously complicated operator (use your imagination). The model variables contain only the dimensions of length and time. The Buckingham Pi theorem tells us that there could be an equivalent physical law containing as few as nine dimensionless variables. Here we will perform the nondimensionalization using the thermal diffusion time H^2/κ as the time scale, the depth H as the length scale, and $-N^2 H$ as the buoyancy scale. Here we will denote dimensionless variables with an asterisk:

$$(u, w) = \frac{\kappa}{H}(u^*, w^*), \tag{9}$$

$$(x, z) = H(x^*, z^*), \tag{10}$$

$$b = -N^2 H b^*, \tag{11}$$

$$\varphi' = \frac{\kappa^2}{H^2}\varphi'^*, \tag{12}$$

$$t = \frac{H^2}{\kappa} t^*. \tag{13}$$

Substitution into (1)-(7) gives:

$$\frac{Du^*}{Dt^*} = -\frac{\partial\,\varphi'^*}{\partial\,x^*} + \frac{\nu}{\kappa}\nabla^{*2} u^*, \tag{14}$$

$$\frac{Dw^*}{Dt^*} = -\frac{\partial\,\varphi'^*}{\partial\,z^*} - \frac{N^2 H^4}{\kappa^2} b'^* + \frac{\nu}{\kappa}\nabla^{*2} w^*, \tag{15}$$

$$\frac{Db'^*}{Dt^*} = w^* + \nabla^{*2} b'^*, \tag{16}$$

and

$$\frac{\partial\,w^*}{\partial\,z^*} = -\frac{\partial\,u^*}{\partial\,x^*}, \tag{17}$$

where

$$\nabla^{*2} \equiv \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial z^{*2}}. \tag{18}$$

If we now define two dimensionless parameters, the Prandtl number

$$P \equiv \frac{\nu}{\kappa} \tag{19}$$

and the Rayleigh number

$$R \equiv \frac{-N^2 H^4}{\nu\kappa}, \tag{20}$$

then (14)and (15) can be written as

$$\frac{Du^*}{Dt^*} = -\frac{\partial\,\varphi'^*}{\partial\,x^*} + P\nabla^{*2} u^* \tag{21}$$

and

$$\frac{Dw^*}{Dt^*} = -\frac{\partial\,\varphi'^*}{\partial\,z^*} + RPb'^* + P\nabla^{*2} w^*. \tag{22}$$

We have succeeded in reducing the model to an equivalent dimensionless system of form

$$\Phi(b'^*, u^*, w^*, \varphi'^*, P, R, x^*, z^*, t^*) = 0, \tag{23}$$

which has 9 dimensionless variables. We are now in a position to efficiently explore properties of solutions to the model by considering the dependence on R and P rather than N^2, H, κ and ν. Unfortunately our notation has become somewhat cumbersome because of the proliferation of asterisks (*). As is often done in this situation, we drop the asterisks and leave them as implied. The dimensionless model is now:

$$\frac{Du}{Dt} = -\frac{\partial\,\varphi'}{\partial\,x} + P\nabla^2 u, \tag{24}$$

$$\frac{Dw}{Dt} = -\frac{\partial\,\varphi'}{\partial\,z} + RPb' + P\nabla^2 w, \tag{25}$$

$$\frac{Db'}{Dt} = w + \nabla^2 b' \tag{26}$$

and

$$\frac{\partial\,w}{\partial\,z} = -\frac{\partial\,u}{\partial\,x}. \tag{27}$$

Although equations (24)-(27) are no longer cumbersome, they now may be more confusing. For example, we have to remember that u really means $u*$. If we wanted to know the dimensional value of u for a configuration with certain values of H and κ, we must use (9) to obtain the value.

Since the flow is incompressible, we can represent it with a stream function with

$$u = \frac{-\partial\psi}{\partial z} \tag{28}$$

and

$$w = \frac{\partial\psi}{\partial x}. \tag{29}$$

The horizontal vorticity can be defined:

$$\omega \equiv \frac{\partial\, w}{\partial\, x} - \frac{\partial\, u}{\partial\, z} = \nabla^2\psi. \tag{30}$$

The vorticity equation derived from (24)-(25) is

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) = RP\frac{\partial b'}{\partial x} + P\nabla^4\psi, \tag{31}$$

and (26) can be written as

$$\frac{\partial b'}{\partial t} + J(\psi, b') = \frac{\partial\psi}{\partial x} + \nabla^2 b', \tag{32}$$

where the Jacobian is defined

$$J(\psi, q) \equiv \frac{\partial\psi}{\partial x}\frac{\partial q}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial q}{\partial x}. \tag{33}$$

2 Convective instability with constant temperature boundaries

In the first example of convective instability we consider constant temperature boundaries so that $b = 0$ on the boundaries at $z = 0$ and $z = 1$. We also consider free-slip boundaries with

$$\frac{\partial^2\psi}{\partial\, z^2} = 0. \tag{34}$$

The rigid condition on the boundary is $w = 0$ or $\psi = constant$. The additional constraint of no vertical averaged lateral flow requires us to take the constant to be the same on both $z = 0$ and $z = 1$; here we use

$$\psi = 0. \tag{35}$$

Linearization of (31) and (32) involves merely dropping the Jacobians. We propose (make a "lucky" guess) a solution consistent with these boundary conditions:

$$\psi = A(t)\sin(\pi z)\sin(kx) \tag{36}$$

and

$$b' = B(t)\sin(\pi z)\cos(kx). \tag{37}$$

The two coupled and linearized p.d.e.'s become two coupled o.d.e.'s for A and B:

$$(\pi^2 + k^2)\left[\frac{d}{dt} + P(\pi^2 + k^2)\right]A(t) = RPkB(t) \tag{38}$$

$$\left[\frac{d}{dt} + (\pi^2 + k^2)\right]B(t) = kA(t) \tag{39}$$

The above two equations can be combined to form a linear, second order equation with constant coefficients:

$$(\pi^2 + k^2)\left[\frac{d}{dt} + P(\pi^2 + k^2)\right]\left[\frac{d}{dt} + (\pi^2 + k^2)\right]A = RPk^2A. \tag{40}$$

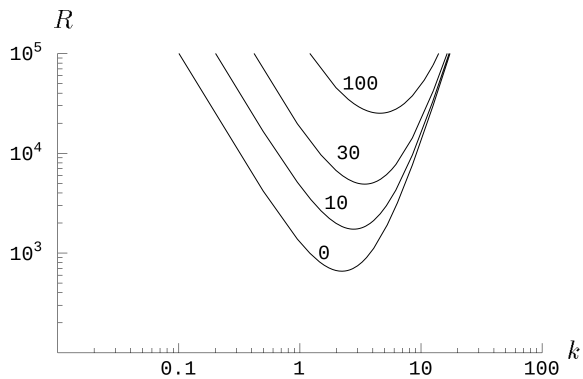

Figure 1: The Rayleigh number R required to produce a growth rate σ as a function of k. The value of σ is indicated. $P = 1$.

Assuming exponential solutions, or

$$\frac{dA}{dt} = \sigma A \tag{41}$$

and non-trivial (nonzero) values of A, the growth rate σ must satisfy

$$(\pi^2 + k^2)\left[\sigma + P(\pi^2 + k^2)\right]\left[\sigma + (\pi^2 + k^2)\right] = RPk^2. \tag{42}$$

A plot of $R(k;\sigma)$ is shown in Fig. 1 for $P = 1$. The marginally stable solutions ($\sigma = 0$) occur at a critical Rayleigh number R_m that is a function of wavenumber only:

$$R_m = \frac{\left(\pi^2 + k^2\right)^3}{k^2}. \tag{43}$$

Clearly $\sigma > 0$ for $R > R_m$ and $\sigma < 0$ for $R < R_m$. We define a critical wavenumber with

$$\frac{\partial\, R_m(k_c)}{\partial\, k} = 0. \tag{44}$$

In this case

$$k_c = \pi/\sqrt{2}. \tag{45}$$

If R is slowly increased, the layer first becomes unstable at the minimum value of R_m, or the critical Rayleigh number

$$R_c \equiv R_m(k_c) = \frac{27}{4}\pi^4. \tag{46}$$

We now seek the physical basis for the wavenumber "preference" in the stability analysis (by "preference" we mean the tendency for wavenumbers near k_c to grow the fastest). The momentum equations can be written as

$$\frac{Du_i}{Dt} = -\frac{\partial\,\varphi}{\partial\, x_i} + RPb'\delta_{i3} + P\nabla^2 u_i. \tag{47}$$

We define a vertical averaging operator as

$$\langle f\rangle = \int_0^1 f\,dz, \tag{48}$$

and a horizontal averaging operator as

$$\overline{f} = \frac{1}{\lambda}\int_0^\lambda f\,dx, \tag{49}$$

where λ is the period of the solution (assuming it is periodic). Multiplying (47) by u_i and applying both averaging operators gives

$$\underset{\text{inertia}}{\frac{\partial}{\partial t}\frac{1}{2}\left\langle\overline{u^2+w^2}\right\rangle} = \underset{\substack{\text{buoyant}\\ \text{production}}}{RP\left\langle\overline{b'w}\right\rangle} \underset{\substack{\text{dissipation}\\ \text{due to } w}}{+P\left\langle\overline{w\nabla^2 w}\right\rangle} \underset{\substack{\text{dissipation}\\ \text{due to } u}}{+P\left\langle\overline{u\nabla^2 u}\right\rangle}. \tag{50}$$

Now, restricting the analysis to the linear solutions where

$$b' = \frac{w}{(\sigma+\pi^2+k^2)}, \tag{51}$$

$$\left\langle\overline{u^2}\right\rangle = \pi^2\left\langle\overline{\psi^2}\right\rangle, \tag{52}$$

$$\left\langle\overline{w^2}\right\rangle = k^2\left\langle\overline{\psi^2}\right\rangle \tag{53}$$

and

$$\frac{d}{dt}\left\langle\overline{\psi^2}\right\rangle = 2\sigma\left\langle\overline{\psi^2}\right\rangle, \tag{54}$$

we have exactly (42), but with the physical processes for each term (or factor) now "tagged" with a process:

$$\underset{\text{inertia}}{\frac{\sigma}{k^2}(\pi^2+k^2)} = \underset{\substack{\text{buoyant}\\ \text{production}}}{\frac{RP}{(\sigma+\pi^2+k^2)}} \underset{\substack{\text{dissipation}\\ \text{due to } w}}{-P(\pi^2+k^2)} \underset{\text{due to } u}{-P(\pi^2+k^2)\frac{\pi^2}{k^2}}. \tag{55}$$

When $\sigma = 0$ we have

$$0 = \frac{R_m}{(\pi^2+k^2)} - (\pi^2+k^2) - (\pi^2+k^2)\frac{\pi^2}{k^2}. \tag{56}$$

At large wavenumber k, horizontal diffusion diminishes b' and the buoyant production decreases. Horizontal diffusion also increases the dissipation of the narrowly spaced updraughts and downdraughts. Thus as $k \to \infty$, $R_m \sim k^4$. At small wavenumber the amplitude of b is limited by diffusion into the boundaries and becomes independent of k, but dissipation increases due to the vertical shear in the horizontal velocity field. Thus as $k \to 0$, $R_m \sim k^2$.

3 A low-order nonlinear model

We will now seek approximate nonlinear solutions to (31) and (32) and propose

$$\psi = A(t)\sin(\pi z)\sin(kx) \tag{57}$$

and

$$b' = B(t)\sin(\pi z)\cos(kx) + C(t)\sin(2\pi z) \tag{58}$$

The nonlinear advection of vorticity turns out to be zero in this solution:

$$J(\psi, \nabla^2\psi) = 0. \tag{59}$$

The advection of b' turns out to be

$$J(\psi, b') = \frac{1}{2}\pi k A B\sin(2\pi z) + \pi k A C\cos(kx)\left[\sin(3\pi z) - \sin(\pi z)\right]. \tag{60}$$

Collecting like Fourier components in (31) and (32) we have

$$\frac{dA}{dt} = \frac{RPk}{(\pi^2+k^2)}B - P\left(\pi^2+k^2\right)A, \tag{61}$$

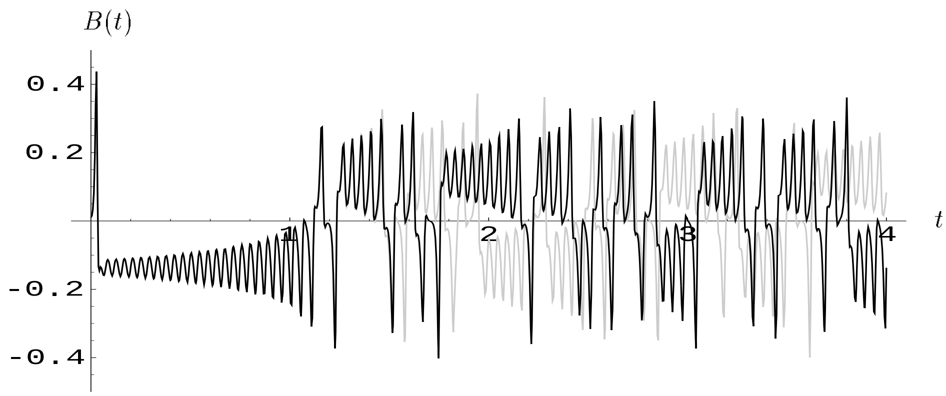

Figure 2: Two solutions for $B(t)$ with $R=28$, $P=10$, and $k=\sqrt{2}\pi/2$. The final time is $t=4$. The initial value $B(0)$ was different by one part in 10^{10}.

$$\frac{dB}{dt} = \pi kAC + kA - \left(\pi^2 + k^2\right) B, \tag{62}$$

and

$$\frac{dC}{dt} = -\frac{\pi k}{2} AB - 4\pi^2 C, \tag{63}$$

except for the extra term proportional to $\sin(3\pi z)$, which is dropped. Notice the nonlinear term containing AB. Equations (61)-(63) have a trivial steady solution $A = B = C = 0$ for all values of R, and two steady, nontrivial solutions for $R > R_c$:

$$A = \pm\frac{\sqrt{8}\sqrt{R - R_c}}{(\pi^2 + k^2)}, \tag{64}$$

$$B = \pm\frac{\sqrt{8}\left(\pi^2 + k^2\right)\sqrt{R - R_c}}{kR}, \tag{65}$$

$$C = \frac{(R_c - R)}{\pi R}. \tag{66}$$

Equations (61)-(63) are actually the famous *Lorenz equations* of chaos theory. Figure 2 shows some chaotic solutions for $B(t)$. See the **Mathematica** notebook for more details.

4 Complete nonlinear model

Here we will seek solutions to (31)-(32) of form

$$\psi(x, z, t) = \sum \Psi_j(t) \sin(m_j \alpha x) \sin(n_j \beta z) \tag{67}$$

and

$$b(x, z, t) = \sum \Theta_j(t) \cos(m_j \alpha x) \sin(n_j \beta z). \tag{68}$$

Here m_j and n_j will be non-negative integers and α and β will be the fundamental wavenumbers of the convection. For a dimensionless layer depth of 1, we will use $\beta = \pi$. The model is "complete" only if we retain an infinite number of modes. For practical applications (as in numerical solutions) we ignore modes with wavenumbers larger than a specified maximum. (In fact, we did that sort of truncation already in the previous section.) The virtue of using a Fourier series for ψ and b' is that the terms are orthogonal; we will be able to derive coupled, first-order nonlinear equations for $\Theta_j(t)$ and $\Psi_j(t)$ (a so-called "autonomous system").

Substitution of (67)-(68) into (31) gives:

$$\frac{\partial}{\partial t}\sum_j \Psi_j(t)(-m_j^2\alpha^2 - n_j^2\beta^2)\sin(m_j\alpha x)\sin(n_j\beta z) =$$
$$-\alpha\beta\sum_j \Psi_j(t)m_j\cos(m_j\alpha x)\sin(n_j\beta z)\sum_k \Psi_k(t)(-m_k^2\alpha^2 - n_k^2\beta^2)\sin(m_k\alpha x)n_k\cos(n_k\beta z)$$
$$+\alpha\beta\sum_j \Psi_j(t)\sin(m_j\alpha x)n_j\cos(n_j\beta z)\sum_k \Psi_k(t)(-m_k^2\alpha^2 - n_k^2\beta^2)m_k\cos(m_k\alpha x)\sin(n_k\beta z)$$
$$-RP\alpha\sum_k \Theta_k(t)m_j\sin(m_k\alpha x)\sin(n_k\beta z)$$
$$+P\sum_j \Psi_j(t)(m_j^2\alpha^2 + n_j^2\beta^2)^2\sin(m_j\alpha x)\sin(n_j\beta z). \tag{69}$$

Now when we multiply every term by $\sin(m_i\alpha x)\sin(n_i\beta z)$ and integrate over one wavelength, only one mode survives in the linear terms:

$$(-m_i^2\alpha^2 - n_i^2\beta^2)\frac{d\Psi_i}{dt} =$$
$$-\alpha\beta\sum\sum \Psi_j(t)\Psi_k(t)(-m_k^2\alpha^2 - n_k^2\beta^2)\gamma_\Psi(m_i, n_i, m_j, n_j, m_k, n_k)$$
$$-RP\sum \Theta_j(t)m_j\delta(m_j, m_i)\delta(n_j, n_i)$$
$$+P\Psi_i(t)(m_i^2\alpha^2 + n_i^2\beta^2)^2, \tag{70}$$

where $\delta(a,b) = 1$ when $a = b$ and $\delta(a,b) = 0$ otherwise. The nonlinear interaction coefficient γ_Ψ is:

$$\gamma_\Psi(m_i, n_i, m_j, n_j, m_k, n_k) =$$
$$m_j n_k \frac{\overline{\langle \sin(m_i\alpha x)\sin(n_i\beta z)\cos(m_j\alpha x)\sin(n_j\beta z)\sin(m_k\alpha x)\cos(n_k\beta z)\rangle}}{\langle \sin(m_i\alpha x)\sin(n_i\beta z)\rangle}$$
$$-n_j m_k \frac{\overline{\langle \sin(m_i\alpha x)\sin(n_i\beta z)\sin(m_j\alpha x)\cos(n_j\beta z)\cos(m_k\alpha x)\sin(n_k\beta z)\rangle}}{\langle \sin(m_i\alpha x)\sin(n_i\beta z)\rangle} \tag{71}$$

In order to determine the value of $\gamma_\Psi(m_i, n_i, m_j, n_j, m_k, n_k)$, we need to express some of the products of the trig functions as sums of trig functions using:

$$\begin{aligned}
&\cos(m_j\alpha x)\sin(n_j\beta z)\sin(m_k\alpha x)\cos(n_k\beta z)\\
= \; &\frac{1}{2}\left(\sin\left[(m_j + m_k)\,\alpha x\right] - \sin\left[(m_j - m_k)\,\alpha x\right]\right)\\
&\frac{1}{2}\left(\sin\left[(n_j + n_k)\,\beta z\right] + \sin\left[(n_j - n_k)\,\beta z\right]\right)\\
= \; &\frac{1}{4}\sin\left[(m_j + m_k)\,\alpha x\right]\sin\left[(n_j + n_k)\,\beta z\right] - \frac{1}{4}\sin\left[(m_j - m_k)\,\alpha x\right]\sin\left[(n_j + n_k)\,\beta z\right]\\
&+\frac{1}{4}\sin\left[(m_j + m_k)\,\alpha x\right]\sin\left[(n_j - n_k)\,\beta z\right] - \frac{1}{4}\sin\left[(m_j - m_k)\,\alpha x\right]\sin\left[(n_j - n_k)\,\beta z\right]
\end{aligned} \tag{72}$$

and

$$\begin{aligned}
&\sin(m_j\alpha x)\cos(n_j\beta z)\cos(m_k\alpha x)\sin(n_k\beta z)\\
= \; &\frac{1}{2}\left(\sin\left[(m_j + m_k)\,\alpha x\right] + \sin\left[(m_j - m_k)\,\alpha x\right]\right)\\
&\frac{1}{2}\left(\sin\left[(n_j + n_k)\,\beta z\right] - \sin\left[(n_j - n_k)\,\beta z\right]\right)\\
= \; &\frac{1}{4}\sin\left[(m_j + m_k)\,\alpha x\right]\sin\left[(n_j + n_k)\,\beta z\right] + \frac{1}{4}\sin\left[(m_j - m_k)\,\alpha x\right]\sin\left[(n_j + n_k)\,\beta z\right]\\
&-\frac{1}{4}\sin\left[(m_j + m_k)\,\alpha x\right]\sin\left[(n_j - n_k)\,\beta z\right] - \frac{1}{4}\sin\left[(m_j - m_k)\,\alpha x\right]\sin\left[(n_j - n_k)\,\beta z\right].
\end{aligned} \tag{73}$$

With the use of (72) and (73) in (71), we have:

$$
\begin{aligned}
&\gamma_\Psi(m_i,n_i,m_j,n_j,m_k,n_k) = \\
&\frac{m_j n_k}{4}\left[\delta(m_j+m_k,m_i) - \delta(m_j-m_k,m_i) + \delta(m_k-m_j,m_i)\right] \\
&\left[\delta(n_j+n_k,n_i) + \delta(n_j-n_k,n_i) - \delta(n_k-n_j,n_i)\right] \\
&-\frac{n_j m_k}{4}\left[\delta(m_j+m_k,m_i) + \delta(m_j-m_k,m_i) - \delta(m_k-m_j,m_i)\right] \\
&\left[\delta(n_j+n_k,n_i) - \delta(n_j-n_k,n_i) + \delta(n_k-n_j,n_i)\right]
\end{aligned}
\tag{74}
$$

As an example of what the nonlinear interaction coefficient γ_Ψ represents, we note that $\gamma_\Psi(2,2,3,1,1,3) = 2$, because $\sin(3\alpha x)\cos(1\beta z)\cos(1\alpha x)\sin(3\beta z)$ gives back some $\sin(2\alpha x)\cos(2\beta z)$, and a bunch of other stuff as well. But $\gamma_\Psi(2,2,3,1,1,5) = 0$; $\sin(3\alpha x)\cos(1\beta z)\cos(1\alpha x)\sin(5\beta z)$ does not produce any $\sin(2\alpha x)\cos(2\beta z)$.

Onwards to the buoyancy equation. Substitution of (67)-(68) into (32) gives:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\sum_j \Theta_j(t)\cos(m_j\alpha x)\sin(n_j\beta z) = \\
&-\sum_j \Psi_j(t) m_j \cos(m_j\alpha x)\sin(n_j\beta z)\sum_k \Theta_k(t)\cos(m_k\alpha x) n_k\cos(n_k\beta z) \\
&-\sum_j \Psi_j(t)\sin(m_j\alpha x) n_j\cos(n_j\beta z)\sum_k \Theta_k(t) m_k\sin(m_k\alpha x)\sin(n_k\beta z) \\
&+\sum_j \Psi_j(t) m_j\cos(m_j\alpha x)\sin(n_j\beta z). \\
&-\sum_j \Theta_j(t)(m_j^2+n_j^2)\cos(m_j\alpha x)\sin(n_j\beta z).
\end{aligned}
\tag{75}
$$

In order to sort out the mess in the nonlinear terms, we need:

$$
\begin{aligned}
&\cos(m_j\alpha x)\sin(n_j\beta z)\cos(m_k\alpha x)\cos(n_k\beta z) \\
=\;& \frac{1}{2}\left(\cos\left[(m_j+m_k)\alpha x\right] + \cos\left[(m_j-m_k)\alpha x\right]\right) \\
& \frac{1}{2}\left(\sin\left[(n_j+n_k)\beta z\right] + \sin\left[(n_j-n_k)\beta z\right]\right) \\
=\;& \frac{1}{4}\cos\left[(m_j+m_k)\alpha x\right]\sin\left[(n_j+n_k)\beta z\right] + \frac{1}{4}\cos\left[(m_j-m_k)\alpha x\right]\sin\left[(n_j+n_k)\beta z\right] \\
& +\frac{1}{4}\sin\left[(m_j+m_k)\alpha x\right]\sin\left[(n_j-n_k)\beta z\right] + \frac{1}{4}\sin\left[(m_j-m_k)\alpha x\right]\sin\left[(n_j-n_k)\beta z\right]
\end{aligned}
\tag{76}
$$

and

$$
\begin{aligned}
&\sin(m_j\alpha x)\cos(n_j\beta z)\sin(m_k\alpha x)\sin(n_k\beta z) \\
=\;& \frac{1}{2}\left(-\cos\left[(m_j+m_k)\alpha x\right] + \cos\left[(m_j-m_k)\alpha x\right]\right) \\
& \frac{1}{2}\left(\sin\left[(n_j+n_k)\beta z\right] - \sin\left[(n_j-n_k)\beta z\right]\right) \\
=\;& -\frac{1}{4}\cos\left[(m_j+m_k)\alpha x\right]\sin\left[(n_j+n_k)\beta z\right] + \frac{1}{4}\cos\left[(m_j-m_k)\alpha x\right]\sin\left[(n_j+n_k)\beta z\right] \\
& +\frac{1}{4}\cos\left[(m_j+m_k)\alpha x\right]\sin\left[(n_j-n_k)\beta z\right] - \frac{1}{4}\cos\left[(m_j-m_k)\alpha x\right]\sin\left[(n_j-n_k)\beta z\right].
\end{aligned}
\tag{77}
$$

This time we get a nonlinear interaction coefficient of:

$$
\begin{aligned}
&\gamma_\Theta(m_i,n_i,m_j,n_j,m_k,n_k) = \\
&\frac{m_j n_k}{4\left[1+\delta(0,m_i)\right]}\left[\delta(m_j+m_k,m_i) + \delta(m_j-m_k,m_i) + \delta(m_k-m_j,m_i)\right]
\end{aligned}
$$

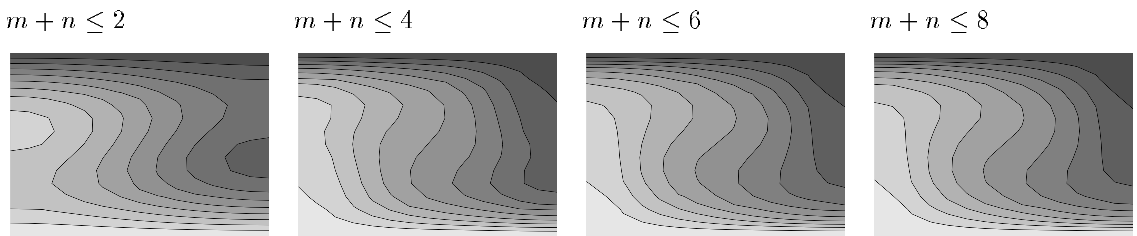

Figure 3: One-half of a wavelength of $b\,(x,z)$ in the asymptotically steady state ($t=4$) for $R=4R_c$, $P=10$, $\alpha=\frac{\sqrt{2}}{2}\pi$, and $\beta=\frac{\sqrt{2}}{2}\pi$. The level of spectral truncation is indicated.

$$
\begin{aligned}
&[\delta(n_j+n_k,n_i)+\delta(n_j-n_k,n_i)-\delta(n_k-n_j,n_i)]\\
&+\frac{n_j m_k}{4\,[1+\delta(0,m_i)]}\,[-\delta(m_j+m_k,m_i)+\delta(m_j-m_k,m_i)+\delta(m_k-m_j,m_i)]\\
&[\delta(n_j+n_k,n_i)-\delta(n_j-n_k,n_i)+\delta(n_k-n_j,n_i)]
\end{aligned}
\tag{78}
$$

for use in:

$$
\begin{aligned}
&\frac{d\Theta_i}{dt}=\\
&-\alpha\beta\sum\sum\Psi_j(t)\Theta_k(t)\gamma_\Theta(m_i,n_i,m_j,n_j,m_k,n_k)\\
&+\alpha\sum_j\Psi_j(t)m_j\delta(m_j,m_i)\delta(n_j,n_i)\\
&-\Theta_i(t)(\alpha^2 m_i^2+\beta^2 n_i^2).
\end{aligned}
\tag{79}
$$

Notice the factor $[1+\delta(0,m_i)]$ in (78). Do you know why it's there?

In Figure 3 we show some results of the spectral model as programmed in **Mathematica**. Various truncations with $m+n$ less than a specified maximum (and $m+n$ being even) are shown. Notice the results converge; there is not much change past $m+n\leq 6$. Some more profound things are said about these solutions within the **Mathematica** notebook.