The Eady model for baroclinic instability

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Baroclinic instability in a thermal wind profile is the most important instability in the atmosphere. We again analyze the same zonal wind, with vertical shear, as in the analysis of symmetric instability. But here we will look at waves (and unstable waves in particular) that develop along this "jet stream". For simplicity, we consider wave disturbances in which $\frac{\partial}{\partial y} = 0$. We will assume the hydrostatic approximation, and therefore restrict the validity of the analysis to wavelengths much greater than the depth of the layer. We will also make a quasi-geostrophic approximation for v', and thus implicitly restrict the growth rate to be much less than f. This model for baroclinic instability, with the assumption of a zonal wind with constant vertical shear, constant N^2, and the quasi-geostrophic and Boussinesq approximation, originates form George Eady (1949).

1 The linearized equations

The appropriate linearized equations are:

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right) b' = -N^2 w' + f\Lambda v', \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right) u' = -\Lambda w' + f v' - \frac{\partial \varphi'}{\partial x}, \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right) v' = -f u', \tag{3}$$

$$b' = \frac{\partial \varphi'}{\partial z}, \tag{4}$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0. \tag{5}$$

We have written

$$\Lambda \equiv \frac{du_o}{dz} \tag{6}$$

and assumed $u_0 = 0$ where $z = 0$. The energy equation from (1)-(6) is:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left\langle\overline{u'^2}\right\rangle + \frac{1}{2}\left\langle\overline{v'^2}\right\rangle + \frac{1}{2N^2}\left\langle\overline{b'^2}\right\rangle\right] = -\Lambda\left\langle\overline{u'w'}\right\rangle + \frac{f\Lambda}{N^2}\left\langle\overline{v'b'}\right\rangle. \tag{7}$$

We define the following nondimensional variables with an asterisk for substitution into (1)-(5):

$$x = L_R x^* = \frac{NH}{f} x^*, \tag{8}$$

$$z = H z^*, \tag{9}$$

$$t = \frac{L_R}{\Lambda H} t^* = \frac{N}{\Lambda f} t^*, \tag{10}$$

$$\varphi' = L_R f N H \varphi'^* = N^2 H^2 \varphi'^*, \tag{11}$$

$$u' = \Lambda H u'^*, \tag{12}$$

$$v' = f L_R v'^* = N H v'^*, \tag{13}$$

$$w' = \frac{\Lambda H^2}{L_R} w'^* = \frac{\Lambda f H}{N} w'^*, \tag{14}$$

and

$$b' = L_R f N b'^* = N^2 H b'^*, \tag{15}$$

where

$$L_R \equiv \frac{NH}{f}. \tag{16}$$

This procedure may seem a bit mysterious. When doing a linear stability analysis, the nondimensionalization helps to keep the notation neat and tidy and also may help in the scale analysis involved in further approximation. In a nonlinear simulation, the nondimensionalization is much more important and allows for efficient exploration of the parameter space. Upon substitution into the governing equations, the asterisks are "dropped" and their existence is left as implied in the subsequent notation. In other words, the variables and equations that follow are dimensionless. (Beware: this custom may be confusing until you become accustomed to it!). Equations (1)-(5) become

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right) b' = -w' + v', \tag{17}$$

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right) u' = -w' + Rv' - R\frac{\partial \varphi'}{\partial x}, \tag{18}$$

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right) v' = -u', \tag{19}$$

$$b' = \frac{\partial \varphi'}{\partial z}, \tag{20}$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{21}$$

where the Richardson number R is defined as:

$$R \equiv \frac{N^2}{\Lambda^2}.$$

In the limit of large R, it appears we can make the quasi- geostrophic approximation and assume v' is exactly geostrophic:

$$v' = \frac{\partial \varphi'}{\partial x}. \tag{22}$$

(Note that u' is not assumed to be geostrophic, it is in fact entirely ageostrophic). Formal justification of (22) requires saying more than just that R is large; we assume that all the variables in (18) are the same order of magnitude. Happily, this turns out to be true in the formal solution of the quasi-geostrophic modes. (We expect this will be so because the coefficients in the governing equations are unity). The vorticity equation derived from (19), (21) and (22) is

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right) \frac{\partial^2 \varphi'}{\partial x^2} = \frac{\partial w'}{\partial z}. \tag{23}$$

The buoyancy equation (17), with (20) and (22) becomes

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right) \frac{\partial \varphi'}{\partial z} = -w' + \frac{\partial \varphi'}{\partial x}. \tag{24}$$

Eliminating w' between (23) and (24), we get

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right) \left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial z^2}\right) = 0. \tag{25}$$

2 The separable solutions

We seek separable solutions to (25) of form

$$\varphi' = \widehat{\varphi}(z)e^{ik(x-ct)}. \tag{26}$$

In separable solutions, if k and c are to be independent if z, we must have

$$\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial z^2} = 0. \tag{27}$$

The top and bottom boundary conditions to apply to (27) follow from (24):

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\frac{\partial \varphi'}{\partial z} - \frac{\partial \varphi'}{\partial x} = 0. \tag{28}$$

One form of the solution to (27) is

$$\widehat{\varphi}(z) = A\cosh(kz) + B\sinh(kz). \tag{29}$$

We apply flat, rigid boundaries at $z = -1/2$ and $z = 1/2$. In seeking the eigenvalue c that allows (28) to be satisfied, it is convenient to define:

$$p \equiv \cosh\left(k/2\right) \tag{30}$$

and

$$q \equiv \sinh\left(k/2\right). \tag{31}$$

The boundary conditions from (28) then define an eigenvalue problem for c:

$$\left(\frac{1}{2} - c\right)k\left(Aq + Bp\right) - \left(Ap + Bq\right) = 0 \tag{32}$$

$$\left(-\frac{1}{2} - c\right)k\left(-Aq + Bp\right) - \left(Ap - Bq\right) = 0. \tag{33}$$

Taking the sum and difference of (32) and (33) makes them a bit tidier:

$$\left(kq - 2p\right)A - 2kcpB = 0 \tag{34}$$

$$-2kcqA + \left(kp - 2q\right)B = 0. \tag{35}$$

The dispersion relation that allows for nontrivial solutions to (34) and (35) is

$$c^2 = \frac{1}{4k^2}\left(k - 2\frac{p}{q}\right)\left(k - 2\frac{q}{p}\right), \tag{36}$$

or

$$\left(kc\right)^2 = \omega^2 = \left[\frac{k}{2} - \tanh\left(\frac{k}{2}\right)\right]\left[\frac{k}{2} - \coth\left(\frac{k}{2}\right)\right]. \tag{37}$$

A plot of ω^2 appears in Fig. 1.

The minimum value of ω^2, and hence maximum value of ω_i, can be obtained from

$$\frac{\partial \ln\left(\omega^2\right)}{\partial k} = \frac{d\ln\left[\frac{k}{2} - \tanh\left(\frac{k}{2}\right)\right]}{dk} + \frac{d\ln\left[\frac{k}{2} - \coth\left(\frac{k}{2}\right)\right]}{dk} = 0 \tag{38}$$

which gives

$$\frac{k}{2}\left[\tanh^2\left(\frac{k}{2}\right) + \coth^2\left(\frac{k}{2}\right)\right] - \left[\tanh\left(\frac{k}{2}\right) + \coth\left(\frac{k}{2}\right)\right] = 0 \tag{39}$$

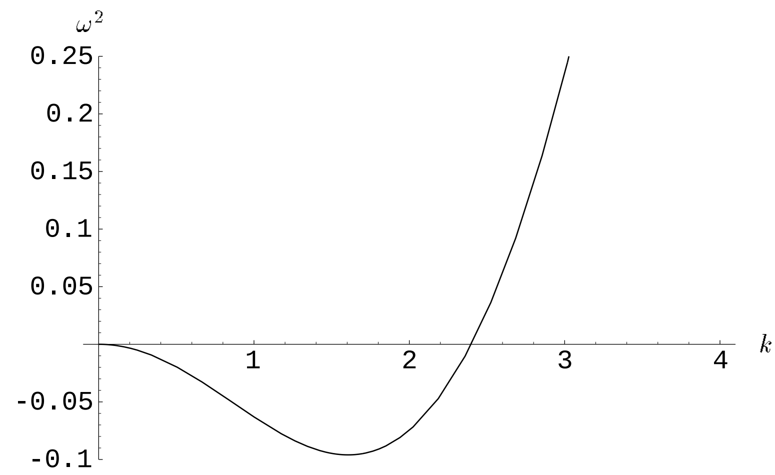

Figure 1: ω^2 of a two-dimensional Eady wave from (37).

and has a solution $k = 1.6062$. Using that value in (37) gives the maximum growth rate:

$$\omega_i = 0.3098. \tag{40}$$

The eigenfunctions v', b', $u'$and w' are easily obtained from φ' using (19)-(22). The eigenfunctions for the most unstable mode appear in Fig. 2 "Eady".

It is an interesting fact that in a semi-infinite atmosphere (either $z > -1/2$ or $z < 1/2$), baroclinic instability does not occur. For an atmosphere unbounded from above, a finite solution is

$$\widehat{\varphi} = Ae^{-kz} \tag{41}$$

and (28) applied as a boundary condition at $z = -1/2$ gives

$$c = \frac{1}{k} - \frac{1}{2}. \tag{42}$$

Modes with $k \to \infty$ are trapped near the lower boundary. The phase of such waves moves with the basic state wind, which at the lower boundary is $u_0(-1/2) = -1/2$. Modes with smaller values of k have $c > -1/2$ and propagate upwind relative to the surface wind, as if they "feel" a steering current higher in the atmosphere. However there is something more profound about the way these waves propagate. The notion of a steering current may be convenient, but it also may be nonsense. The eigenfunctions for this neutral baroclinic waves are plotted in Fig. 2 "Bottom" for the case of $k = 2$, which gives $c = 0$ so the wave is stationary in the frame of the viewer. Notice that despite the shear, the phase of the waves remains vertical, rather than being tilted as some passive quantity would be. The wave propagates to the right, relative to the fluid, for $z < 0$, and to the left, relative to the fluid, for $z > 0$.

Consider how how relative vorticity $\frac{\partial^2 \varphi'}{\partial x^2}$ is maintained in steady state in the viewer's reference frame. From (23) with $c = 0$ (or $\frac{\partial}{\partial t} = 0$) this balance is

$$z \frac{\partial}{\partial x} \frac{\partial^2 \varphi'}{\partial x^2} = \frac{\partial w'}{\partial z}. \tag{43}$$

The northerly geostrophic wind advects cold air southward. The cold air tends to sink during the process of geostrophic adjustment, but the condition $w = 0$ at $z = -1/2$ must still be satisfied; notice that $\frac{\partial w}{\partial z}$ changes sign at $z = 0$. Planetary vorticity is being compressed below $z = 0$ where it is being stretched above. Thus the stretching term in the vorticity equation can exactly balance the advection term, which also changes sign at $z = 0$. We have now "explained" how vorticity can remain stationary with respect to the viewer. We can also "explain" how the buoyancy remains stationary with respect to the viewer. The northerly wind causes cooling while the associated subsidence of the geostrophic adjustment causes warming. Below $z = 0$ the cooling effect of the northerly wind is larger, while above $z = 0$ the heating effect of the subsidence is larger. The horizontal advection term changes sign at $z = 0$ and balances the change in sign of the net buoyancy tendency from the other terms. The reader is invited to explain how any of the other fields remain stationary.

For an atmosphere unbounded from below, a finite solution is

$$\widehat{\varphi} = Ae^{kz}. \tag{44}$$

and (28) applied as a boundary condition at $z = +1/2$ gives

$$c = -\frac{1}{k} + \frac{1}{2} \tag{45}$$

The eigenfunctions for this mode are plotted in Fig. 2 "Top", with the phase shifted by a quarter wavelength relative to the mode in Fig. 2 "Bottom". The reason for considering this phase shift will be explained below.

One heuristic model for baroclinic instability is that, in the presence of bounding surfaces at both the top and the bottom, the neutral waves will lock on to each other and cause instability if their phase speeds match. This condition is known as *phase locking*. From (42) and (45), phase locking occurs if $k = 2$. Is it only a coincidence that this value of k is near the value that gives the maximum growth rate in the Eady model, or is there really something profound in this relationship? Fig. 2 "Top + Bottom" shows a superposition of the neutral waves of Fig. 2 "Bottom" and Fig. 2 "Top". Of course the neutral waves, either individually or in sum, are no longer really solutions to the equations because they do not satisfy both rigid boundary conditions: notice $w \neq 0$ on the boundaries. Nevertheless, the superposition of the waves has a striking similarity to the unstable mode of Fig. 2 "Eady". The dimensionless version of (7) is

$$\frac{\partial}{\partial t}\left\langle \overline{\frac{1}{2}\left(\frac{u'^2}{R} + v'^2\right) + \frac{1}{2}b'^2} \right\rangle = -\frac{1}{R}\left\langle \overline{u'w'} \right\rangle + \left\langle \overline{v'b'} \right\rangle. \tag{46}$$

At large R the first term on the right becomes insignificant. In fact, in quasi-geostrophic energetics it is exactly zero.

Notice from inspection of Fig. 2 "Top" and Fig. 2 "Bottom" how the energy source term is zero for either neutral wave considered by itself. Now let the lower boundary wave be denoted by a subscript "1" and the upper boundary wave be denoted by a subscript "2". When the waves interact , or $v = v_1 + v_2$ and $b = b_1 + b_2$, the energy source is

$$\left\langle \overline{v'b'} \right\rangle = \left\langle \overline{v'_1 b'_1} \right\rangle + \left\langle \overline{v'_2 b'_2} \right\rangle + \left\langle \overline{v'_1 b'_2} \right\rangle + \left\langle \overline{v'_2 b'_1} \right\rangle \tag{47}$$

and the two self-interaction terms are again zero. When we displace the phase of the upper wave a quarter wavelength to the left and add the upper wave and lower wave together as in Fig. 2 "Top + Bottom", the mutual interaction terms are both positive. Phase locking is thus able to tap into the energy stored in the north-south temperature gradient and produces instability.

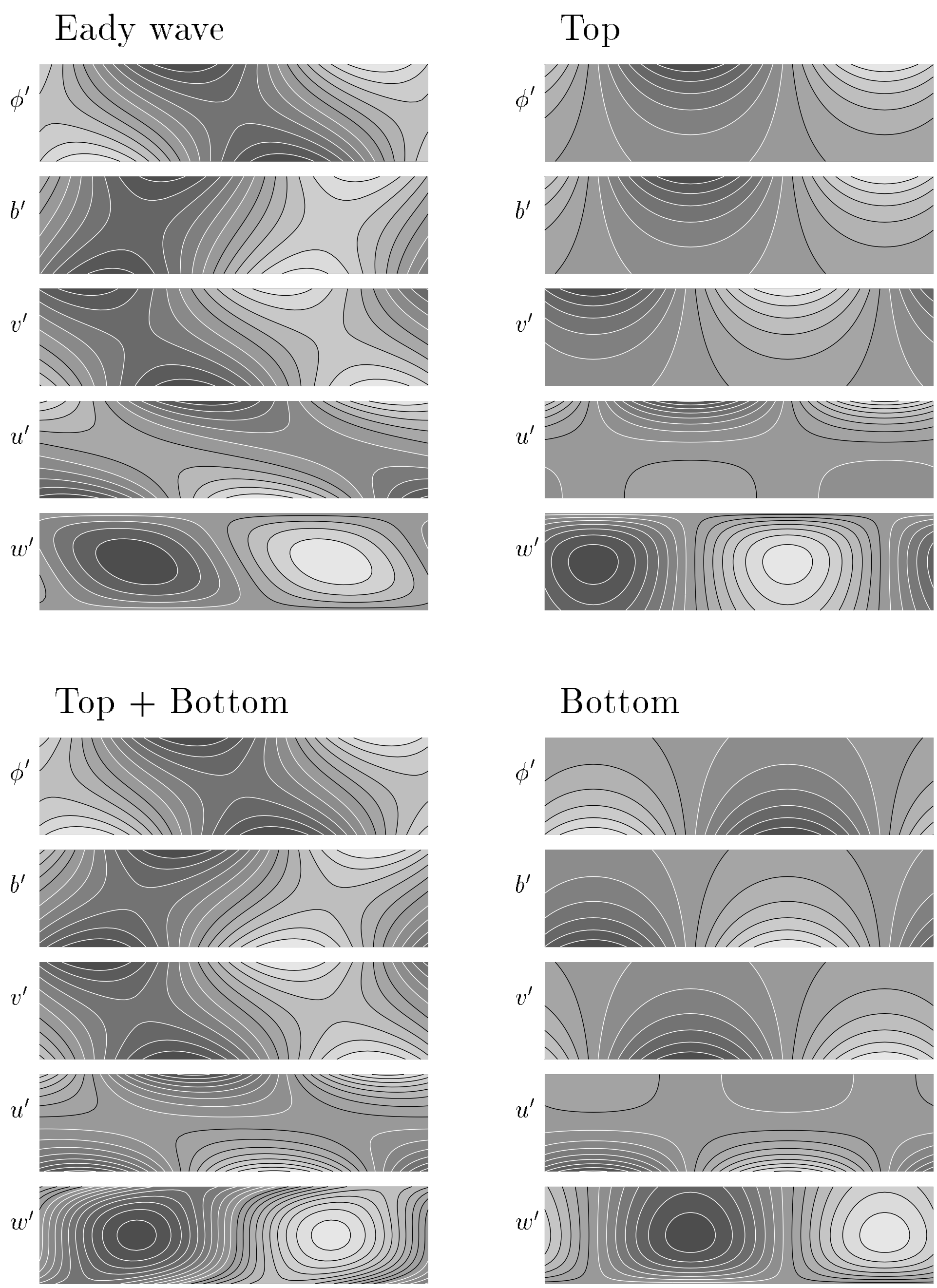

Figure 2: Baroclinic waves. Contour intervals for ϕ', b', v', u' and w' are .3, .6, .6, .5, and .2. White contours are negative, black contours are positive.