Anelastic Equations for the Atmosphere

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1 Declaration of assumptions

We hold these truths about ideal gases to be self-evident:

$$p = R\rho T \tag{1}$$

and

$$dq = c_p dT - \frac{1}{\rho} dp. \tag{2}$$

If we define a potential temperature θ with

$$\frac{dq}{T} \equiv c_p d\ln(\theta), \tag{3}$$

then (2) becomes

$$c_p d\ln(\theta) = c_p d\ln(T) - R d\ln(p) \tag{4}$$

which integrates to

$$T = \theta \left(\frac{p}{p_o}\right)^{R/c_p} \tag{5}$$

where p_o is an arbitrary constant "reference" value of pressure. It is convenient to define an Exner function:

$$\pi \equiv \left(\frac{p}{p_o}\right)^{R/c_p} \tag{6}$$

so

$$T = \theta\pi, \tag{7}$$

$$\rho = \frac{p_o}{R\theta}\pi^{c_v/R} \tag{8}$$

and

$$p = p_o\pi^{c_p/R}. \tag{9}$$

The evolution of θ in a parcel is governed by (3), so θ is relatively easy to prognosticate. In many important applications parcel motion is adiabatic, or in other words diabatic effects are absent ($d\theta = 0$) and $D\theta/Dt = 0$. If we now restrict consideration to this important case, the continuity equation can be expressed as

$$\frac{D\rho\theta}{Dt} + \rho\theta\nabla\cdot\vec{U} = 0 \tag{10}$$

or

$$\frac{D\pi^{c_v/R}}{Dt} + \pi^{c_v/R}\nabla\cdot\vec{U} = 0. \tag{11}$$

An important consequence of the Exner function is

$$\frac{1}{\rho}\nabla p = c_p\theta\nabla\pi. \tag{12}$$

The governing equations can now be expressed in terms of θ and π rather than ρ and p, If we make the f-plane or β-plane approximation to the atmosphere, the inviscid adiabatic momentum equations can be written as:

$$\frac{\partial}{\partial t}\pi^{c_v/R} + \nabla \cdot \left(\pi^{c_v/R}\vec{U}\right) = 0 \tag{13}$$

$$\frac{D\theta}{Dt} = 0 \tag{14}$$

$$\frac{Du}{Dt} - fv = -c_p\theta\frac{\partial \pi}{\partial x} \tag{15}$$

$$\frac{Dv}{Dt} + fu = -c_p\theta\frac{\partial \pi}{\partial y} \tag{16}$$

$$\frac{Dw}{Dt} = -c_p\theta\frac{\partial \pi}{\partial z} - g \tag{17}$$

In many situations, especially analysis of stability, it is convenient to write the variables as the sum of a steady, basic state (subscript o) and a departure from that state ($\prime$):

$$\pi(x, y, z, t) \equiv \pi_o(x, y, z) + \pi'(x, y, z, t) \tag{18}$$

and so on for the other variables. If we assume the basic state is geostrophic, then

$$fu_o \equiv -c_p\theta_o\frac{\partial \pi_o}{\partial y} \tag{19}$$

and

$$fv_o \equiv c_p\theta_o\frac{\partial \pi_o}{\partial x} \tag{20}$$

With $w_o \equiv 0$, the basic state is in hydrostatic balance:

$$c_p\theta_o\frac{\partial \pi_o}{\partial z} = -g. \tag{21}$$

Equations (15) - (17) become

$$\frac{Du}{Dt} - fv' = -fv_o\frac{\theta'}{\theta_o} - c_p\theta\frac{\partial \pi'}{\partial x} \tag{22}$$

$$\frac{Dv}{Dt} + fu' = fu_o\frac{\theta'}{\theta_o} - c_p\theta\frac{\partial \pi'}{\partial y} \tag{23}$$

and

$$\frac{Dw'}{Dt} = -c_p\theta\frac{\partial \pi'}{\partial z} + g\frac{\theta'}{\theta_o} \tag{24}$$

Here we try to clean up these equations by showing that in certain regimes certain terms will be small. Assume the height scale for a disturbance is H and the horizontal length scale is L. If we assume that departures from basic state are in nearly hydrostatic balance, then scale analysis indicates that

$$c_p\theta\,|\pi'| = \frac{gH}{\theta}\,|\theta'| \tag{25}$$

Therefore the first term on the r.h.s. of (22) can be neglected compared to the second term on the r.h.s. if

$$fLv_o << gH \tag{26}$$

A similar approximation can be made in (23). If we also assume

$$L\frac{\partial \theta_o}{\partial x} << \theta_o \tag{27}$$

and

$$H\frac{\partial\,\theta_o}{\partial\,z} << \theta_o \tag{28}$$

then there exists a constant such that

$$|\theta - \theta_{oo}| << \theta_o. \tag{29}$$

We next replace θ by θ_{oo} except in the derivatives of θ. Also, in order to keep our equations neat and tidy, we have use for

$$\varphi' \equiv c_p\theta_{oo}\pi' \tag{30}$$

$$b \equiv g\frac{\theta}{\theta_{oo}}. \tag{31}$$

Equations (14)-(17) become

$$\frac{Db}{Dt} = 0 \tag{32}$$

$$\frac{Du}{Dt} - fv' = -\frac{\partial\,\varphi'}{\partial\,x} \tag{33}$$

$$\frac{Dv}{Dt} + fu' = -\frac{\partial\,\varphi'}{\partial\,y} \tag{34}$$

and

$$\frac{Dw'}{Dt} = -\frac{\partial\,\varphi'}{\partial\,z} + b' \tag{35}$$

We next derive the *thermal wind relations* for the basic state. From (19) and (21)

$$\frac{\partial}{\partial\,z}\left(\frac{fu_o}{c_p\theta_o}\right) = -\frac{\partial^2\,\pi_o}{\partial\,y\partial\,z} = \frac{\partial}{\partial\,y}\left(\frac{g}{c_p\theta_o}\right) \tag{36}$$

or

$$f\frac{\partial\,u_o}{\partial\,z} - \frac{fu_o}{\theta_o}\frac{\partial\,\theta_o}{\partial\,z} = -\frac{g}{\theta_o}\frac{\partial\theta_o}{\partial\,y}. \tag{37}$$

Assuming

$$\left|\frac{\partial\ln\,|u_o|}{\partial\,z}\right| >> \left|\frac{\partial\,\ln|\theta_o|}{\partial\,z}\right| \tag{38}$$

we have

$$f\frac{\partial\,u_o}{\partial\,z} = -\frac{g}{\theta_o}\frac{\partial\theta_o}{\partial\,y} \tag{39}$$

which can be further approximated as

$$f\frac{\partial\,u_o}{\partial\,z} = -\frac{\partial\,b_o}{\partial\,y}. \tag{40}$$

Similarly, from (19) and (20)

$$f\frac{\partial\,v_o}{\partial\,z} = \frac{\partial\,b_o}{\partial\,x}. \tag{41}$$

We also introduce the definition

$$N^2 \equiv \frac{\partial\,b_o}{\partial\,z} \tag{42}$$

Assuming no veering or backing of the basic state thermal wind, so that in the basic state $\partial b/\partial t = 0$, then $Db/Dt = 0$ gives

$$\frac{Db'}{Dt} = -\frac{Db_o}{Dt} = -u'\frac{\partial\,b_o}{\partial\,x} - v'\frac{\partial\,b_o}{\partial y} - w'\frac{\partial\,b_o}{\partial\,z} \tag{43}$$

or

$$\frac{Db'}{Dt} = -N^2w' - f\frac{\partial\,v_o}{\partial\,z}u' + f\frac{\partial\,u_o}{\partial\,z}v'. \tag{44}$$

We have not said anything yet about the prime quantities being small. We do so now in order to make the anelastic approximation to the mass continuity equation, which we write as

$$\frac{\partial}{\partial t}\pi^{c_v/R} = -\nabla \cdot \left(\pi^{c_v/R}\vec{U}\right) \tag{45}$$

Except in sound waves, the terms on the right nearly balance. Since buoyancy effects in our model equations are determined solely by the advection of θ or b, we are allowed to be sloppy in modelling the small residual mass convergence in the the velocity field that must be associated with local p changes. We instead require

$$\nabla \cdot \left(\pi_o^{c_v/R}\vec{U}\right) = 0 \tag{46}$$

As a result, the model equations will not allow for sound waves, but the modelling of meteorological phenomenon will be relatively unaffected. Using (8), (46) can also be written as:

$$\nabla \cdot \left(\rho_o\theta_o\vec{U}\right) = 0 \tag{47}$$

In the atmosphere, ρ_o varies much more with height than θ_o does. Traditionally, (47) has been further approximated as:

$$\nabla \cdot \left(\rho_o\vec{U}\right) = 0 \tag{48}$$

To make a long story short, we will give many of our examples in an atmosphere with a rigid lid at height L, much less than the scale height of the atmosphere. The slight variation of ρ_o over the depth of the layer can then be ignored and the anelastic approximation becomes:

$$\nabla \cdot \vec{U} = 0 \tag{49}$$

With the above "closure" for the problem, the model equations are exactly those for an incompressible fluids ($D\rho/Dt = 0$ and $\nabla \cdot U = 0$) with the application of the Boussinesq approximation, which follows from assuming that $|\rho_o - \rho_{oo} << \rho_{oo}|$ for some ρ_{oo} and taking the density as constant except where the density produces a "buoyancy force". The equations are the same as above, but with $\varphi' = \frac{p'}{\rho_{oo}}$ and $b' = -\frac{g\rho'}{\rho_{oo}}$.

Recapping, the model equations are:

$$\frac{Db'}{Dt} = -N^2 w' - f\frac{\partial v_o}{\partial z}u' + f\frac{\partial u_o}{\partial z}v' \tag{50}$$

$$\frac{Du}{Dt} - fv' = -\frac{\partial \varphi'}{\partial x} \tag{51}$$

$$\frac{Dv}{Dt} + fu' = -\frac{\partial \varphi'}{\partial y} \tag{52}$$

$$\frac{Dw'}{Dt} = -\frac{\partial \varphi'}{\partial z} + b' \tag{53}$$

and one of the following:

$$\nabla \cdot \left(\rho_o\theta_o\vec{U}\right) = 0 \tag{54}$$

$$\nabla \cdot \left(\rho_o\vec{U}\right) = 0 \tag{55}$$

or

$$\nabla \cdot \vec{U} = 0 \tag{56}$$

2 Example: Internal Gravity Waves

Consider small amplitude waves in an atmosphere with $f = 0$, $u_o = 0$ and $v_o = 0$. We consider two dimensional waves in which $v' = 0$ as well. We will work with the form of (55):

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} + \frac{1}{\rho_o}\frac{d\rho_o}{dz}w' = 0. \tag{57}$$

If we let $H(z)$ be the defined as the scale height for density, (57) can be written as

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} - \frac{1}{H}w' = 0. \tag{58}$$

Appropriate for small amplitude waves, the linearized version of (51)-(53) are:

$$\frac{\partial b'}{\partial t} = -N^2 w' \tag{59}$$

$$\frac{\partial u'}{\partial t} = -\frac{\partial \varphi'}{\partial x} \tag{60}$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial \varphi'}{\partial z} + b' \tag{61}$$

We look for solutions of form:

$$u' = \widehat{u}(z)e^{i(kx-\omega t)} \tag{62}$$

$$w' = \widehat{w}(z)e^{i(kx-\omega t)} \tag{63}$$

$$b' = \widehat{b}(z)e^{i(kx-\omega t)} \tag{64}$$

$$\varphi' = \widehat{\varphi}(z)e^{i(kx-\omega t)} \tag{65}$$

$$\frac{d\,\widehat{w}}{d\,z} - \frac{1}{H}\widehat{w} = -ik\widehat{u} \tag{66}$$

$$i\omega\widehat{b} = N^2\widehat{w} \tag{67}$$

$$\omega\widehat{u} = k\widehat{\varphi} \tag{68}$$

$$-i\omega\widehat{w} = -\frac{d\,\widehat{\varphi}}{d\,z} + \widehat{b} \tag{69}$$

Eliminating between (66) and (68):

$$\frac{d\,\widehat{w}}{d\,z} - \frac{1}{H}\widehat{w} = -i\frac{k^2}{\omega}\widehat{\varphi} \tag{70}$$

Eliminating between (67) and (69):

$$\frac{d\,\widehat{\varphi}}{d\,z} = i\omega\widehat{w} - i\frac{N^2}{\omega}\widehat{w} \tag{71}$$

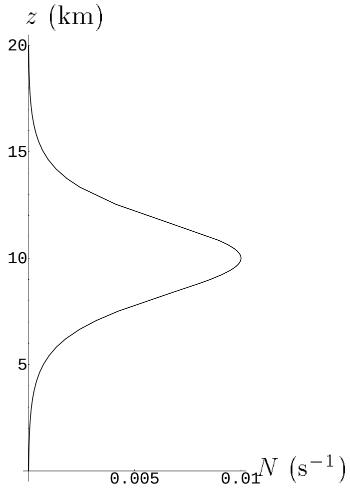
If we eliminate $\widehat{\varphi}$ between (70) and (71):

$$\frac{d^2\widehat{w}}{dz^2} - \frac{1}{H}\frac{d\widehat{w}}{dz} + \frac{1}{H^2}\frac{dH}{dz}\widehat{w} - k^2\left(1 - \frac{N^2}{\omega^2}\right)\widehat{w} = 0 \tag{72}$$

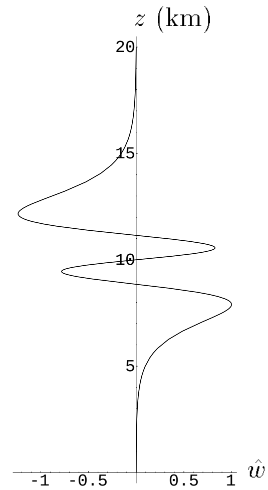
An example of a solution to (72) is shown in Figure 1.

If we assume that H is nearly constant (it is exactly constant in an isothermal atmosphere), we can ignore the third term in (72). If we further assume that N^2 is a constant, we will have a constant coefficient equation:

$$\frac{d^2\widehat{w}}{dz^2} - \frac{1}{H}\frac{d\widehat{w}}{dz} - k^2\left(1 - \frac{N^2}{\omega^2}\right)\widehat{w} = 0. \tag{73}$$



Profile of $N(z) = .01\text{sech}^2\left[8\left(\frac{z}{z_{lid}} - \frac{1}{2}\right)\right]$.



Profile of $\hat{w}(z)$.

Figure 1: A Solution of (72) for $\hat{w}(z)$ with $H = 10$ km and $N(z)$ as specified on the left. The frequency is $\omega = .003249$. The solution was obtained using **Mathematica**. The wave is "trapped" in the stable layer.

A solution of form

$$\widehat{w}(z) = e^{\alpha z} \tag{74}$$

requires that

$$\alpha^2 - \frac{\alpha}{H} - k^2\left(1 - \frac{N^2}{\omega^2}\right) = 0 \tag{75}$$

or

$$\alpha = \frac{1}{2H} \pm \sqrt{\frac{1}{4H^2} + k^2\left(1 - \frac{N^2}{\omega^2}\right)}. \tag{76}$$

The above can be written as

$$\alpha = \frac{1}{2H} \pm im \tag{77}$$

where

$$m^2 = -\frac{1}{4H^2} - k^2\left(1 - \frac{N^2}{\omega^2}\right) \tag{78}$$

The dispersion relation is

$$\omega^2 = \frac{N^2}{m^2 + k^2 + \frac{1}{4H^2}} \tag{79}$$

Assuming $m^2 > 0$, the general solution for $\hat{w}$ is

$$\widehat{w} = Ae^{z/2H}\cos(mz) + Be^{z/2H}\sin(mz) \tag{80}$$

A rigid lower boundary at $z = 0$ requires that $A = 0$. If m is large enough, say to satisfy a rigid lid requirement at $z = z_{lid} << H$, then we can ignore the terms with H and work with

$$\omega^2 = \frac{k^2N^2}{m^2 + k^2} \tag{81}$$

and

$$\widehat{w} = B\sin(mz). \tag{82}$$

We find solutions for the other fields, in the limit $\frac{1}{H} \to 0$, using (66), (67) and (68):

$$\widehat{u} = iB\frac{m}{k}\cos(mz) \tag{83}$$

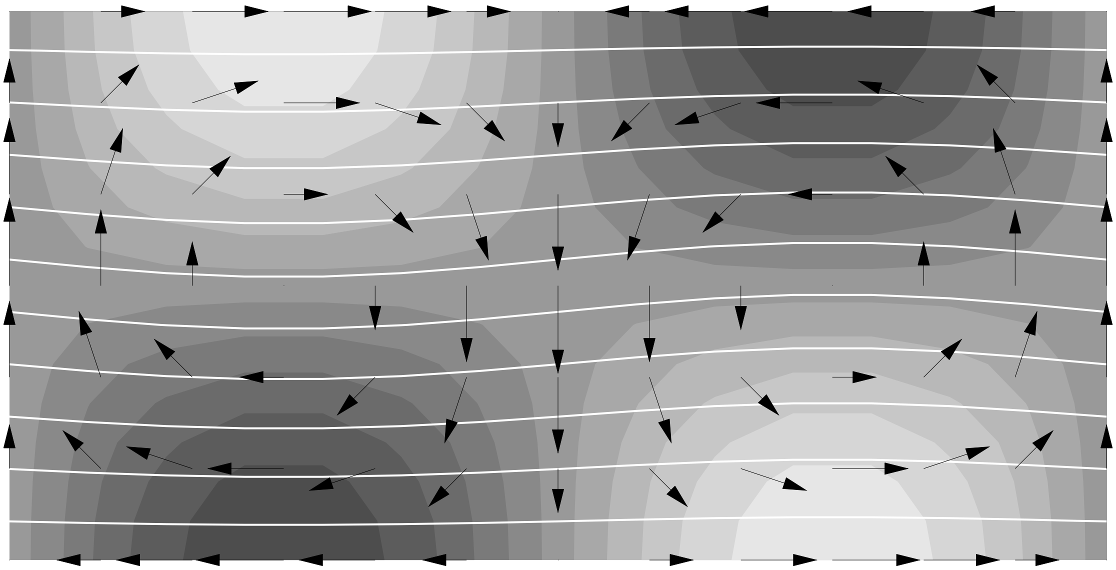


Figure 2: Vertical cross-section of an internal wave bounded by rigid boundaries at the top and bottom. The white lines are the total buoyancy field b. The gray is pressure field ϕ'; dark is low pressure. The vectors are the (u, w) velocity field. The wave is travelling to the right. Plotted with **Mathematica** using (90)-(93)

$$\widehat{\varphi} = iB\frac{\omega m}{k^2}\cos(mz) \tag{84}$$

$$\widehat{b} = -i\frac{N^2}{\omega}B\sin(mz). \tag{85}$$

Assuming B is real, the real parts of the fields are:

$$w' = Bsin(mz)\cos(kx - \omega t) \tag{86}$$

$$u' = -B\frac{m}{k}\cos(mz)\sin(kx - \omega t) \tag{87}$$

$$\varphi' = -B\frac{\omega m}{k^2}\cos(mz)\sin(kx - \omega t) \tag{88}$$

$$b' = \frac{N^2}{\omega}B\sin(mz)\sin(kx - \omega t) \tag{89}$$

Assuming B and ω are real, and letting $\alpha \equiv \frac{k}{m}$ and $B = C\omega$

$$\frac{w'}{C\,|\omega|} = \pm sin(mz)\cos(kx - \omega t) \tag{90}$$

$$\frac{u'}{C\,|\omega|} = \mp\frac{1}{\alpha}\cos(mz)\sin(kx - \omega t) \tag{91}$$

$$\frac{\varphi' m}{N^2 C} = \frac{-1}{1 + \alpha^2}\cos(mz)\sin(kx - \omega t) \tag{92}$$

$$\frac{b'}{N^2 C} = \sin(mz)\sin(kx - \omega t) \tag{93}$$

where the $\pm$ or $\mp$ are associated with the positive and negative root for ω respectively.